UNIVERSITY OF GEORGIA

DEPARTMENT OF MATHEMATICS
MATH 2250 - FINAL EXAM
SPRING 2025

PRINTED NAME : Solutions

STUDENT ID : ______

DATE : ___/___/___

GRADE
/150

INSTRUCTOR : ______ CLASS TIME : ______

INSTRUCTIONS

№	SCORE	MAX
1		5
2		5
3		5
4		5
5		5
6		5
7		5
8		5
9		5
10		5
11		5
12		5
13		5
14		5
15		5
16		5
17		5
18		5
19		16
20		10
21		24
22		10
TOTAL		150

- The exam lasts 3 hours and it has two parts: the first one consists of Multiple Choice (MC) questions, and the second part of Free Response (FR) ones. You must show work for both parts, unless explicitly told otherwise. An unjustified answer will receive no credit. If you are using a shortcut, explain it.
- Your work must be neat and organized. Circle the answer for MC questions and put a box around the final answer for the FR questions. There is only one correct choice for each MC question.
- Smart devices, including smart watches and cell phones, are prohibited and must not be within reach.
- If you plan to use a calculator, only TI-30XS MultiView (the name must match exactly) is permitted; no other calculators or sharing of calculators is allowed.
- Provide an exact answer for each problem. Answers containing symbolic expressions such as $\cos(3)$ and $\ln(2)$ are perfectly acceptable, but $\sin(\pi/2) = 1$.
- If additional space is needed, use the last two pages. Write "cont'd" (continued) in the designated area and continue on the scrap paper by first writing the problem number and then continuing your solution. Work outside the specified area without any indication, will not be graded.

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Part I: Multiple Choice

Show work and circle the final answer.

1. [5 pts] If $y = 2x^3 + 3x^2 - 12x - 10$, find $y'(-1)$.

(A) -18
(B) -12
(C) -6
(D) 0
(E) 6

$y'(x) = 6x^2 + 6x - 12$
$y'(-1) = 6(-1)^2 + 6(-1) - 12$
$= -12$ //

_____ PTS

2. [5 pts] If $f(\theta) = \sin\theta$, find $f^{(11)}(\pi)$, the 11th derivative of $f(\theta)$ at $\theta = \pi$.

(A) $-\pi$
(B) -1
(C) 0
(D) 1
(E) π

$f(\theta) = \sin\theta$
$f'(\theta) = \cos\theta$
$f''(\theta) = -\sin\theta$
$f'''(\theta) = -\cos\theta$
$f^{IV}(\theta) = \sin\theta$
So $f^{(8)}(\theta) = f^{(IV)}(\theta) = \sin\theta$
and $f^{(11)}(\theta) = f^{(3)}(\theta) = -\cos\theta$
At $\theta = \pi$, $-\cos\theta = 1$
Thus $f^{(11)}(\pi) = 1$ //

_____ PTS

3. [5 pts] How many critical points of $y = x^5 - \frac{20}{3}x^3$ are NOT local extrema?

(A) 0
(B) 1
(C) 2
(D) 3
(E) 5

domain of y : $(-\infty, \infty)$

$y' = 5x^4 - 20x^2$

critical points :

$y' = 5x^2(x^2-4) = 0 \Rightarrow x = 0, -2, 2.$

c.p		-2		0		2	
sign(f')	$+$	$\circ$	$-$	$\circ$	$-$	$\circ$	$+$
		loc. max.				loc. min.	

Only $x=0$ is not a local extremum, thus the answer is one.

______ PTS

4. [5 pts] If $y(x) = \sqrt{1-x^2} + x \arcsin x$, compute $y'(x)$.

(A) $\dfrac{x}{\sqrt{1-x^2}}$

(B) $\dfrac{1}{\sqrt{1-x^2}}$

(C) $\arcsin x - \dfrac{2x}{\sqrt{1-x^2}}$

(D) $\arcsin x - \dfrac{x}{\sqrt{1-x^2}}$

(E) $\arcsin x$

$y'(x) = \frac{1}{\cancel{2}}(1-x^2)^{-1/2}\cdot(-\cancel{2}x) + x\cdot\frac{1}{\sqrt{1-x^2}} + \arcsin x$

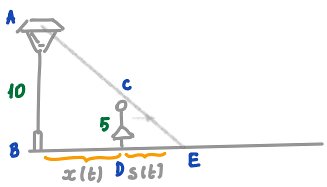
$= \cancel{\frac{-x}{\sqrt{1-x^2}}} + \cancel{\frac{x}{\sqrt{1-x^2}}} + \arcsin x$

$= \arcsin x$ //

______ PTS

5. [5 pts] A person walks away from a lamp post at a rate of 3 ft/sec. The lamp post is 10 ft tall, and the person is 5 ft tall. How fast is the length of the person's shadow increasing when the person is 15 ft from the lamp post?

(A) $\frac{3}{2}$ ft/sec
(B) $\frac{10}{3}$ ft/sec
(C) $\frac{5}{3}$ ft/sec
(D) 2 ft/sec
(E) 3 ft/sec



Let $x(t)$ be the horizontal distance of the person from the lamp post, and $s(t)$ the shadow of the person.

Given: $\frac{dx}{dt} = 3$

Want: $\frac{ds}{dt} = ?$ when $x = 15$.

____ PTS

$\triangle CED \sim \triangle AEB$ so $\frac{5}{10} = \frac{s(t)}{s(t)+x(t)} \Leftrightarrow x(t)+s(t) = 2s(t) \Leftrightarrow x(t) = s(t)$

$\frac{d}{dt}$ $\qquad x'(t) = s'(t) = 3$ $\qquad \frac{d}{dt}$

The length of the person's shadow is increasing at a rate of 3 ft/sec.

6. [5 pts] What is the slope of the tangent line to $xy^2 + e^y = x^2$ at the point $(1, 0)$?

(A) -1
(B) 0
(C) $\frac{1}{2}$
(D) 1
(E) 2

Differentiating both sides with respect to x, we have:

$x \cdot 2yy' + y^2 + y'e^y = 2x$

At $(1,0)$:

$y' \cdot e^0 = 2 \Rightarrow y' = 2$

____ PTS

7. [5 pts] Consider the function

$$f(x) = \begin{cases} x^2 + ax - 3x + 3 - a, & \text{if } x \le 1 \\ \dfrac{2x}{x^2+1}, & \text{if } x > 1 \end{cases}$$

What should a be so that the function is differentiable everywhere?

(A) $a = 0$

(B) $a = \frac{1}{2}$

(C) $a = \frac{1}{3}$

(D) $a = \frac{1}{4}$

(E) $a = 1$

$f(x)$ is a piecewise function and both branches are continuous everywhere with no sharp corners. The only place where $f(x)$ can fail to be differentiable is at $x=1$ so to demand differentiability there, we must demand two things:

______ PTS

(I) continuity at $x=1$; i.e. $f(1) = \underbrace{\lim_{x\to 1} f(x)}_{(*)}$

$(*)$ $\lim_{x\to 1^-} x^2 + ax - 3x + 3 - a = \lim_{x\to 1^+} \frac{2x}{x^2+1}$

true for all a

(II) tangent lines must be the same

$\lim_{x\to 1^-} 2x + a - 3 = \lim_{x\to 1^+} \frac{-2x^2+2}{(x^2+1)^2}$

$a - 1 = 0 \Rightarrow \boxed{a=1}$

8. [5 pts] If $f(x) = x^3 + 8x + \cos(3x)$ and $g(x) = f^{-1}(x)$, find the slope of the tangent line to $g(x)$ at the point $(1, 0)$.

(A) $\frac{1}{11}$

(B) $\frac{1}{8}$

(C) $-\frac{1}{8}$

(D) $\frac{1}{11 + 3\sin 3}$

(E) 8

$[f^{-1}(1)]' = \dfrac{1}{f'(\underbrace{f^{-1}(1)}_{=0})}$ as $f(0)=1$

$= \dfrac{1}{3x^2 + 8 - 3\sin(3x)}\Big|_{x=0}$

$= \dfrac{1}{8}$ //

OR

______ PTS

$f^{-1}(f(x)) = x$

$(f^{-1})'(\underbrace{f(x)}_{1}) \cdot f'(\underset{0}{x}) = 1$

$(f^{-1})'(1) = \dfrac{1}{f'(0)}$

$= \dfrac{1}{3x^2 + 8 - 3\sin(3x)}\Big|_{x=0}$

$= \dfrac{1}{8}$ //

9. [5 pts] Let f and g be differentiable functions, and suppose the following values are known:

$$f(a) = -4, \quad f'(a) = 8, \quad g(a) = c, \quad g'(a) = b,$$
$$f(c) = 3, \quad f'(c) = 2, \quad g(c) = 5, \quad g'(c) = 1.$$

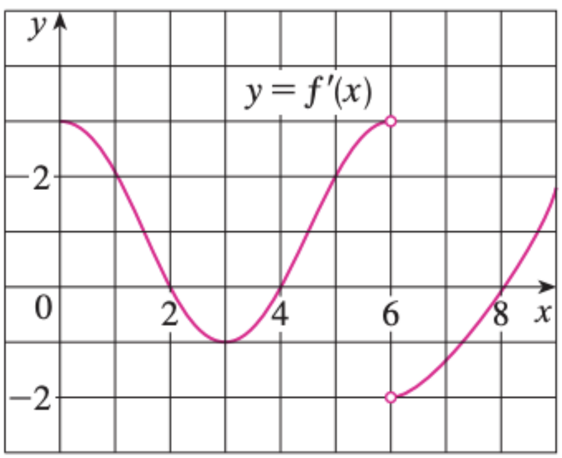
Define $h(x) = [f(g(x))]^2$. What is the value of $h'(a)$?

- (A) $4b$
- (B) $6b$
- (C) $12b$
- (D) $15b$
- (E) $24b$
- (F) $36b$

____ PTS

$$h'(a) = 2\,\overbrace{f(\underbrace{g(a)}_{c})}^{3} \cdot \overbrace{f'(\underbrace{g(a)}_{c})}^{2} \cdot \overbrace{g'(a)}^{b}$$
$$= 12b.$$

10. [5 pts] Below is the graph of the derivative f' of a continuous function f.



At what value(s) of x does f have a local maximum?

- (A) 0 only
- (B) 2 only
- (C) 4 only
- (D) 0 and 6
- (E) 2 and 4
- (F) 4 and 6
- (G) 2 and 6
- (H) 4 and 8

____ PTS

local maximum at $x = a$ $\iff$ $f'(a)$ is 0 or DNE <u>and</u> f' is positive to the left of a, and negative to the right.

The places where the above happens are at $x = 2$ and $x = 6$. //

11. [5 pts] If $H(x) = \int_0^{2x} e^{3t^2}\,dt$, find $H'(1)$.

- (A) e^3
- (B) e^6
- (C) e^{12}
- (D) $2e^6$
- (E) $2e^{12}$

$H'(x) = \frac{d}{dx}\int_0^{2x} e^{3t^2}\,dt = 2e^{3(2x)^2} = 2e^{12x^2}$

$H'(1) = 2e^{12}.$

______ PTS

12. [5 pts] Let $f(x)$ be a continuous function defined for all real numbers. We are given that

$$\int_{-1}^{1} f(x)\,dx = 3, \quad \text{and} \quad \int_{-1}^{2} f(x)\,dx = 5.$$

Calculate $\int_1^2 (3 - 2f(x))\,dx$.

- (A) -1
- (B) 0
- (C) 1
- (D) 5
- (E) 7

$\int_1^2 (3 - 2f(x))\,dx = \int_1^2 3\,dx - 2\int_1^2 f(x)\,dx$

$= 3(2-1) - 2\left[\underbrace{\int_1^{-1} f(x)\,dx}_{-3} + \underbrace{\int_{-1}^{2} f(x)\,dx}_{5}\right]$

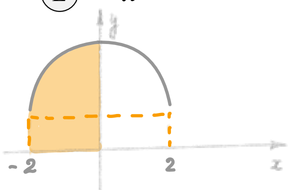
$= 3 - 2\cdot 2$

$= -1$ //

______ PTS

13. [5 pts] Compute $\int_{-2}^{0} 1+\sqrt{4-x^2}\,dx$ by recognizing it as the area of a region made up of simple geometric shapes.

- (A) $\pi + 2$
- (B) $-\pi - 2$
- (C) 2π
- (D) $\pi + 4$
- (E) $-\pi$



$\int_{-2}^{0}\left(1+\sqrt{4-x^2}\right)dx$ = Area of rectangle with base 2 and height 1 + Area of a quarter of a circle with radius 2

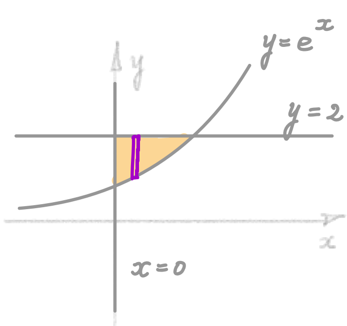
$= 2\cdot 1 + \frac{1}{4}\pi\cdot 2^2$

$= 2 + \pi$ //

______ PTS

14. [5 pts] Compute the area bounded by $y = e^x$, $x = 0$, and $y = 2$.

- (A) $2\ln(2) - 1$
- (B) $2\ln(2) + 1$
- (C) $\ln(2) - 1$
- (D) $2 - \ln(2)$
- (E) $2\ln(2)$



Points of intersection:

$e^x = 2 \Rightarrow x = \ln 2$

Area $= \int_0^{\ln 2} (2 - e^x)\,dx$

$= 2x - e^x \Big|_0^{\ln 2}$

$= (2\ln 2 - 2) - (0 - e^0)$

$= 2\ln 2 - 1$ //

______ PTS

15. [5 pts] A scientist is studying how the concentration of a chemical in a solution changes with temperature. The concentration C, measured in moles per liter, is modeled by the function

$$C(T) = \ln(T^2 + 5)$$

where T is the temperature in degrees Celsius. To make quick predictions without using a calculator, the scientist wants to use a linear approximation of $C(T)$ near $T = 2°C$. Estimate the concentration at $T = 2.1°C$.

(A) $C(2.1) \approx \ln(9) + \dfrac{1}{45}$

(B) $C(2.1) \approx \ln(9) + \dfrac{1}{90}$

(C) $C(2.1) \approx \ln(9) + \dfrac{2}{45}$

(D) $C(2.1) \approx \ln(9) + \dfrac{4}{81}$

(E) $C(2.1) \approx \ln(9) + \dfrac{1}{9}$

$$C(2.1) \approx C(2) + C'(2)(2.1-2)$$

$$= \ln(2^2+5) + \left.\frac{2T}{T^2+5}\right|_{T=2} \cdot 0.1$$

$$= \ln 9 + \frac{\cancel{4}\,2}{9} \cdot \frac{1}{\cancel{10}\,5}$$

$$= \ln 9 + \frac{2}{45} \;//$$

____ PTS

16. [5 pts] $\displaystyle\int_0^2 e^{-x^2}\,dx$ is to be approximated using a Riemann sum with 8 subintervals of equal width, where the height of each rectangle is determined by the right endpoint of the subinterval. If i is an integer between 1 and 8, what is the area of the i^{th} rectangle?

(A) $\dfrac{1}{8}e^{-\frac{i}{4}}$

(B) $\dfrac{1}{8}e^{-\frac{i^2}{4}}$

(C) $\dfrac{1}{4}e^{-\frac{i^2}{4}}$

(D) $\dfrac{1}{4}e^{-\left(\frac{i}{4}\right)^2}$

(E) $e^{-\frac{i}{4}}$

$A_i = f(x_i)\cdot\Delta x$ where $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{8} = \dfrac{1}{4}$ and $x_i = a + i\Delta x = 0 + \dfrac{i}{4} = \dfrac{i}{4}$

$= e^{-\left(\frac{i}{4}\right)^2} \cdot \dfrac{1}{4} \;//$

____ PTS

17. [5 pts] Evaluate $\int_0^1 \frac{x}{1+x^2}\,dx$.

(A) $\ln(2)$

(B) $\frac{1}{2}$

(C) $\frac{1}{4}\ln(2)$

(D) $\arctan(1) = \frac{\pi}{4}$

(E) $\frac{1}{2}\ln(2)$

Let $u = 1 + x^2$ $\quad x = 0 .. 1$

$du = 2x\,dx$ $\quad u = 1 .. 2$

$$\int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{1}{2}\int_1^2 \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u|\Big|_1^2$$

$$= \frac{1}{2}\ln 2 \,//$$

_____ PTS

18. [5 pts] If $f(x)$ is continuous and it is known that $\int_0^2 f(x)\,dx = 6$, evaluate $\int_0^{\frac{\pi}{2}} f(2\sin t)\cos t\,dt$.

(A) 0

(B) $\frac{3}{2}$

(C) 2

(D) 3

(E) 4

Let $x = 2\sin t$ $\quad t = 0 .. \frac{\pi}{2}$

$dx = 2\cos t\,dt$ $\quad x = 0 .. 2$

$$\int_0^{\frac{\pi}{2}} f(2\sin t)\cos t\,dt$$

$$= \frac{1}{2}\underbrace{\int_0^2 f(x)\,dx}_{=6}$$

$$= 3 \,//$$

_____ PTS

Part II: Free Response

Present all work leading to your final answer clearly and in a structured manner. Show all relevant steps and justify them.

19. [16 pts] Compute the following limits:

(a) $\lim_{t \to 1^-} \sqrt{2t+5}$ $= \sqrt{7}$

4 PTS

(b) $\lim_{x \to 2} \frac{x^2 - 7x + 10}{x^2 - 4}$ $\overset{[\frac{0}{0}]}{=} \lim_{x \to 2} \frac{(\cancel{x-2})(x-5)}{(\cancel{x-2})(x+2)} = -\frac{3}{4}$ //

OR by L'Hôpital

$= \lim_{x \to 2} \frac{2x-7}{2x} = -3/4$ //

4 PTS

(c) $\lim_{x \to 1} \frac{3^x - 3}{x - 1}$ $\overset{[\frac{0}{0}]}{=} \lim_{x \to 1} \frac{3^x \ln 3}{1} = 3 \ln 3$ //

by L'Hôpital

4 PTS

(d) $\lim_{y \to +\infty} \sqrt{y^2 + 6y - 1} - y$ $\overset{[\infty - \infty]}{=} \lim_{x \to +\infty} \left(\sqrt{y^2 + 6y - 1} - y\right) \cdot \frac{\left(\sqrt{y^2+6y+1} + y\right)}{\sqrt{y^2+6y+1} + y}$

$= \lim_{x \to +\infty} \frac{\cancel{y^2} + 6y - 1 - \cancel{y^2}}{\sqrt{y^2 + 6y - 1} + y}$

$= \lim_{x \to +\infty} \frac{\overset{3}{\cancel{6}}\cancel{y}}{\cancel{2}\cancel{y}}$

$= 3$ //

4 PTS

20. [10 pts]

(a) Write the limit definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

OR

$$= \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

⋮

______ 2 PTS

(b) Use that definition to compute the derivative of $f(x) = \dfrac{1}{x+5}$. (No credit will be awarded for any other method.)

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h+5} - \frac{1}{x+5}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h+5} - \frac{1}{x+5}\right)$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{\cancel{x+5} - (\cancel{x}+h+\cancel{5})}{(x+h+5)(x+5)}$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \cdot \frac{-\cancel{h}}{(x+h+5)(x+5)}$$

$$= \frac{-1}{(x+5)^2} \quad //$$

We can check our result using chain rule, for example.

$f(x) = (x+5)^{-1}$ and $f'(x) = -(x+5)^{-2}$. //

______ 8 PTS

21. [24 pts] Consider the function

$$f(x) = \frac{x}{x^2+1}$$

with first derivative

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

and second derivative:

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

(a) Find the domain of $f(x)$.

domain of $f(x) := (-\infty, \infty)$

_______ 2 PTS

(b) Find the $x-$ and $y-$ intercepts of $f(x)$.

x-intercepts : set $y=0 \iff x=0$

y-intercepts : set $x=0 \iff y = \frac{0}{0+1} = 0$

$\therefore$ $(0,0)$ is the only point of intersection with the coordinate axes.

_______ 2 PTS

(c) Does $f(x)$ possess any symmetry? (Even/Odd/Neither)

$f(-x) = \frac{-x}{(-x)^2+1} = -\frac{x}{x^2+1} = -f(x)$

$\therefore$ $f(x)$ is odd.

_______ 2 PTS

SUBTOTAL _______ 6 PTS

(d) Does $f(x)$ have any vertical or horizontal asymptotes? Justify.

- There are no vertical asymptotes as the denominator is never zero.
- $\lim_{x \to \pm\infty} \frac{x}{x^2+1} = 0 \quad \therefore \boxed{y=0}$ is the equation of the horizontal asymptote for $f(x)$.

______ 4 PTS

(e) Using the first and/or the second derivative of $f(x)$, find all critical points.

$f'(x) = \frac{1-x^2}{(x^2+1)^2}$ $\stackrel{=0}{\longrightarrow}$ $x = \pm 1$

DNE $\longrightarrow$ never

______ 2 PTS

(f) Using the first and/or the second derivative of $f(x)$, find all inflection points.

$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$ $\longrightarrow$ $x = 0, \pm\sqrt{3}$

$\stackrel{\neq 0}{\longrightarrow}$ never

______ 2 PTS

SUBTOTAL ______ 8 PTS

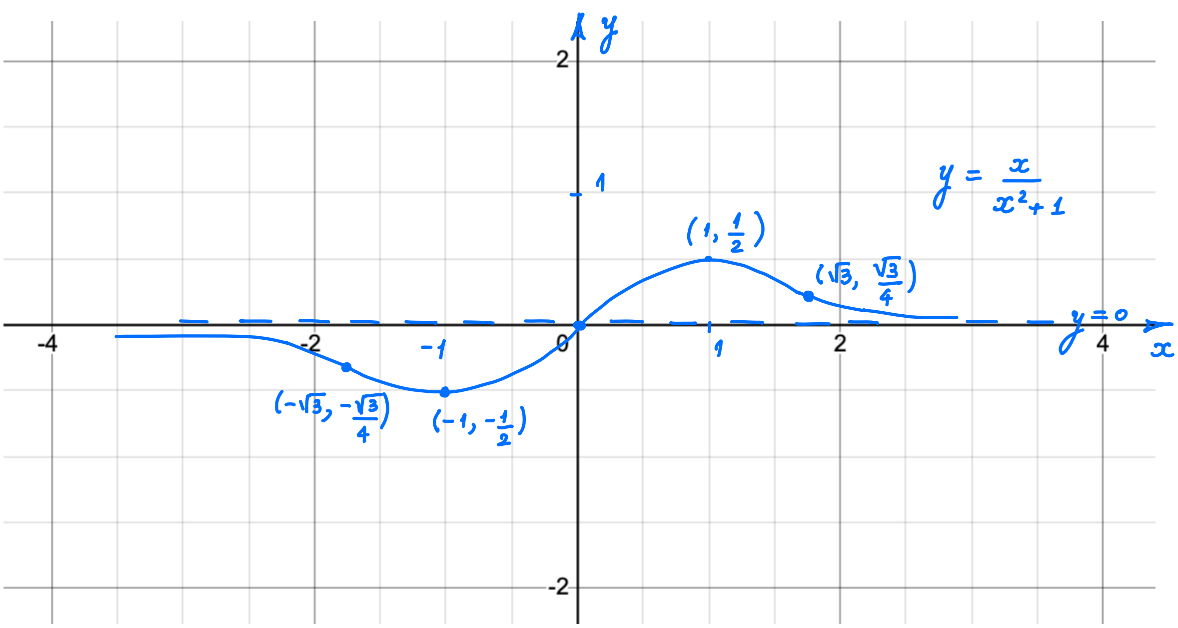
(g) Organize the above findings in a table that encodes the signs of the first and second derivative of $f(x)$. Clearly state the interval(s) where the function is increasing, decreasing, concave up or concave down. If not applicable, state "none".

		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
sign (f')	$-$		$-$	0	$+$		$+$	0	$-$		$-$
sign (f'')	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$

$$f(x) \text{ is } \begin{cases} \text{increasing on } (-1, 1) \\ \text{decreasing on } (-\infty, -1) \cup (1, \infty) \\ \text{concave up on } (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty) \\ \text{concave down on } (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}). \end{cases}$$

______ 6 PTS

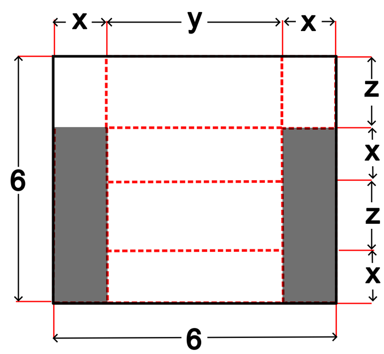
(h) Sketch the function, putting in evidence intercepts, local extrema, inflection points and asymptotes, if any. Do not forget to label your graph.



______ 4 PTS

SUBTOTAL ______ 10 PTS

22. [10 pts] A closed box with a lid is to be formed by trimming identical rectangles (shaded below) from two adjacent corners of a flat, 6-foot by 6-foot cardboard sheet with negligible thickness. The sides are then folded up to form the box. A diagram is provided here:



(a) Write a formula for the volume of the box in terms of x along with a reasonable domain. Simplify the volume formula as much as you can.

$$V = xyz$$
$$= x(6-2x)(3-x)$$
$$= (6x-2x^2)(3-x)$$
$$= 18x - 6x^2 - 6x^2 + 2x^3$$
$$= 18x - 12x^2 + 2x^3 \quad \text{where } x \in (0,3)$$

______ 4 PTS

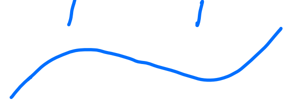
(b) Use calculus to find the value of x that gives the largest volume and justify your finding.

$$V' = 18 - 24x + 6x^2$$
$$= 6(3-4x+x^2)$$
$$= 6(x-1)(x-3)$$

$V' = 0$ at $x = 1$ and $x = 3$.

		1		3	
sign (V')	+	0	−	0	+

Behavior of $V(x)$:



______ 6 PTS

So $V(x)$ reaches a global maximum at $x = 1$. //

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