Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Geometry of Selberg's bisectors in the symmetric space $SL(n,\mathbb{R})/SO(n,\mathbb{R})$

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Abstract

We study several problems about the symmetric space associated with the Lie group $SL(n,\mathbb{R})$. These problems are connected to an algorithm based on Poincaré's Fundamental Polyhedron Theorem, designed to determine generalized geometric finiteness properties for subgroups of $SL(n,\mathbb{R})$. The algorithm is analogous to the original one in hyperbolic spaces, while the Riemannian distance is replaced by an $SL(n,\mathbb{R})$-invariant premetric.

The main results of this article are twofold. In the first part, we focus on questions that occurred in generalizing Poincaré's algorithm to our symmetric space. We describe and implement an algorithm that computes the face-poset structure of finitely-sided polyhedra, and construct an angle-like function between hyperplanes. In the second part, we study further questions related to hyperplanes and Dirichlet-Selberg domains in our symmetric space. We establish several criteria for the disjointness of hyperplanes and classify particular Abelian subgroups of $SL(3,\mathbb{R})$ based on whether their Dirichlet-Selberg domains are finitely-sided or not.

1. *Introduction*

1.1. *Backgrounds*

The space $SL(n,\mathbb{R})/SO(n)$ studied in this paper is the Riemannian symmetric space associated with the Lie group $SL(n,\mathbb{R})$. As a symmetric space of non-compact type ($A_{n-1}\mathrm{I}$) in Cartan's classification[**9**], we consider it a generalization of the hyperbolic space. Using the Killing form on $\mathfrak{sl}(n)$ and the Cartan decomposition of $SL(n,\mathbb{R})$[**4**], one describes the space $SL(n,\mathbb{R})/SO(n)$ as follows:

Definition 1. The *hypersurface model* of $SL(n,\mathbb{R})/SO(n)$ is defined as the set

$$\mathcal{P}(n) = \mathcal{P}_{hyp}(n) = \{X \in Sym_n(\mathbb{R}) \mid \det(X) = 1,\ X > 0\}, \tag{1.1}$$

equipped with the metric tensor

$$\langle A, B\rangle_X = \mathrm{tr}(X^{-1}AX^{-1}B),\ \forall A, B \in T_X\mathcal{P}(n).$$

Here, $Sym_n(\mathbb{R})$ denotes the vector space of $n \times n$ real symmetric matrices, and $X > 0$ (or $X \geq 0$) means that X is positive definite (or positive semi-definite, respectively). Throughout the paper, we consider the bilinear form $\langle A, B\rangle := \mathrm{tr}(A \cdot B)$ on $Sym_n(\mathbb{R})$ and interpret the orthogonality accordingly.

The group $SL(n,\mathbb{R})$ acts on $\mathcal{P}(n)$ as isometries via congruence transformations:

$$SL(n,\mathbb{R}) \curvearrowright \mathcal{P}(n),\ g.X = g^{\mathrm{T}}Xg.$$

We also introduce another model of $\mathcal{P}(n)$:

2000 *Mathematics Subject Classification* 20F65 (primary), 22E40, 53C35(secondary).

Definition 2. The *projective model* of $\mathcal{P}(n)$ is defined as follows:

$$\mathcal{P}(n) = \mathcal{P}_{proj}(n) = \{[X] \in \mathbf{P}(Sym_n(\mathbb{R})) \,|\, X > 0\}. \tag{1.2}$$

It is evident that $\mathcal{P}_{proj}(n)$ and $\mathcal{P}_{hyp}(n)$ are diffeomorphic. The *standard Satake compactification* and *Satake boundary* of $\mathcal{P}(n)$ are defined through the projective model:

Definition 3. The standard Satake compactification of $\mathcal{P}(n)$ is the set

$$\overline{\mathcal{P}(n)}_S = \{[X] \in \mathbf{P}(Sym_n(\mathbb{R})) \,|\, X \geq 0\},$$

and the Satake boundary of $\mathcal{P}(n)$ is the set

$$\partial_S \mathcal{P}(n) = \overline{\mathcal{P}(n)}_S \backslash \mathcal{P}(n).$$

We anticipate that many concepts and methodologies in hyperbolic spaces will have analogs in the symmetric space $\mathcal{P}(n)$. Of particular interest is the generalization of Poincaré's Algorithm, initially proposed by Riley[**19**] for hyperbolic 3-space and extended by Epstein and Petronio [**6**] for hyperbolic n-space, aimed at determining whether a subgroup of $SO^+(n,1)$ is geometrically finite.

Poincaré's Algorithm typically involves constructing Dirichlet domains, which are convex polytopes in hyperbolic space. However, Dirichlet domains in $\mathcal{P}(n)$ appear non-convex and impractical for study. Hence, we adopt an $SL(n,\mathbb{R})$-invariant proposed by Selberg[**21**] as a substitute of the Riemannian distance on $\mathcal{P}(n)$:

Definition 4. For $X, Y \in \mathcal{P}(n)$, the *Selberg's invariant* from X to Y is defined as

$$s(X,Y) = \mathrm{tr}(X^{-1}Y).$$

Selberg's invariant satisfies that $s(X,Y) \geq n$ for any $X, Y \in \mathcal{P}(n)$, with equality if and only if $X = Y$. Consequently, Selberg defines analogs of bisectors and Dirichlet domains:

Definition 5. The *(Selberg's) bisector* of two points $X, Y \in \mathcal{P}(n)$ is defined as

$$Bis(X,Y) = \{Z \in \mathcal{P}(n) | s(X,Z) = s(Y,Z)\}.$$

The *Dirichlet-Selberg domain* for a discrete subgroup $\Gamma < SL(n,\mathbb{R})$ centered at the point $X \in \mathcal{P}(n)$ is defined as

$$DS(X,\Gamma) = \{Y \in \mathcal{P}(n) | s(X,Y) \leq s(g.X,Y),\ \forall g \in \Gamma\}.$$

As in [**11**], Dirichlet-Selberg domains are also defined for discrete subsets of $SL(n,\mathbb{R})$ for computational purposes.

To comprehend the polyhedral nature of Dirichlet-Selberg domains, we generalize to $\mathcal{P}(n)$ the concept of hyperbolic convex polyhedra. For instance, a *d-plane* in $\mathcal{P}(n)$ is defined as the non-empty intersection of $\mathcal{P}(n)$ with a $(d+1)$-dimensional linear subspace of the vector space $Sym_n(\mathbb{R})$. Other notions such as *hyperplanes*, *half-spaces*, and *convex polyhedra* in $\mathcal{P}(n)$, along with *facets*, *ridges*, and *faces* of a convex polyhedron P in $\mathcal{P}(n)$, can be defined analogously to those in the hyperboloid model of hyperbolic spaces [**18**]. We denote the set of facets, ridges, and faces of a convex polyhedron P by $\mathcal{S}(P)$, $\mathcal{R}(P)$ and $\mathcal{F}(P)$, respectively. Additionally, we denote by *span*(P) the minimal plane in $\mathcal{P}(n)$ containing the convex polyhedron P.

The group $SL(n,\mathbb{R})$ acts on planes and convex polyhedra in $\mathcal{P}(n)$, enabling the definition of *fundamental polyhedra* for subgroups of $SL(n,\mathbb{R})$, *exact convex polyhedra* in $\mathcal{P}(n)$, *facet pairings* for exact convex polyhedron P, and *ridge cycles* and the *quotient space* for a facet pairing Φ, analogously to the hyperbolic case, [**18**]. Notably, Dirichlet-Selberg domains for discrete subgroups of $SL(n,\mathbb{R})$ serve as fundamental polyhedra for them:

Proposition 1.1 [**11**]. *For a discrete subgroup $\Gamma < SL(n,\mathbb{R})$ and a point $X \in \mathcal{P}(n)$, the Dirichlet-Selberg domain $DS(X,\Gamma)$ forms a convex polyhedron in $\mathcal{P}(n)$. Moreover, if $Stab_\Gamma(X)$ is trivial, $DS(X,\Gamma)$ serves as a fundamental polyhedron for Γ.*

With these notions established, we describe an analog of Poincaré's Algorithm for $SL(n,\mathbb{R})$:

Poincaré's Algorithm (tentative). Suppose that we have a finite set of elements $\{g_1,\dots,g_n\} \subset SL(n,\mathbb{R})$ and a point $X \in \mathcal{P}(n)$ as the center of Dirichlet-Selberg domains. The following algorithm determines if the subgroup Γ generated by these elements admits a finitely-sided Dirichlet-Selberg domain centered at X:

(1) Start with $l = 1$ and compute the finite subset $\Gamma_l \subset \Gamma$, consisting of elements represented by words of length $\leq l$ in the letters of g_i and g_i^{-1}.
(2) Compute the face poset of the Dirichlet-Selberg domain $DS(X,\Gamma_l)$, which is a finitely-sided polyhedron in $\mathcal{P}(n)$.
(3) Utilizing this face poset data, check if $DS(X,\Gamma_l)$ satisfies the following conditions:
    (i) Verify if $DS(X,\Gamma_l)$ is an exact convex polyhedron. Namely, for each $w \in \Gamma_l$, ensure that the isometry w pairs the two facets contained in $Bis(X, w.X)$ and $Bis(X, w^{-1}.X)$ if they are non-empty.
    (ii) Verify if $DS(X,\Gamma_l)$ satisfies the tiling condition, i.e., if the quotient space M obtained by identifying the paired facets of $DS(X,\Gamma_l)$ is a $\mathcal{P}(n)$-orbifold. We consider formulating this with a "ridge cycle condition".
    (iii) Verify if the quotient space M is complete.
    (iv) Verify if each element g_i is generated by the facet pairings of $DS(X,\Gamma_l)$, following the method provided in [**19**].
(4) If any of these conditions are not satisfied, increment l by 1 and repeat the steps above.
(5) If these conditions are satisfied, Poincaré's Fundamental Polyhedron Theorem and Proposition 1.1 imply that $DS(X,\Gamma_l)$ is a fundamental domain for Γ, analogously to the hyperbolic case [**18**]. Consequently, Γ is a geometrically finite subgroup of $SL(n,\mathbb{R})$. Specifically, Γ is discrete, with a finite presentation derived from the ridge cycles of $DS(X,\Gamma_l)$.

This generalized Poincaré's algorithm prompts several questions, motivating the results discussed in this paper.

1.2. *Preliminaries*

Below we provide the essential preliminaries preceding presenting the main results of this paper. We begin with introducing *co-oriented hyperplanes*:

Definition 6. The normal space of a non-zero matrix $A \in Sym_n(\mathbb{R})$ is defined as

$$A^\perp = \{X \in \mathcal{P}(n) \,|\, \mathrm{tr}(X \cdot A) = 0\},$$

constituting a hyperplane in $\mathcal{P}(n)$ whenever non-empty. We designate A as a *normal vector* of the hyperplane $A^\perp$. A hyperplane associated with a normal vector is called a *co-oriented hyperplane.*

The normal vector of a hyperplane is unique up to a nonzero multiple. Identical co-oriented hyperplanes with normal vectors that differ by a positive multiple are regarded as the same co-oriented hyperplanes. Conversely, identical co-oriented hyperplanes with normal vectors that differ by a negative multiple from each other are said to be oppositely oriented. If σ is a co-oriented hyperplane given by $A^{\perp}$, then the co-oriented hyperplane with the opposite orientation is denoted by $-\sigma$ or $(-A)^{\perp}$.

We define a co-oriented hyperplane σ to *lie between* two co-oriented hyperplanes $A^{\perp}$ and $B^{\perp}$ if the normal vector associated with σ is a positive linear combination of A and B.

Some of our main results rely on *matrix pencils*:

DEFINITION 7. A real (or complex) *matrix pencil* is a set $\{A - \lambda B | \lambda \in \mathbb{R}\}$ (or $\lambda \in \mathbb{C}$, respectively), where A and B are real $n \times n$ matrices. We denote this matrix pencil by (A, B).

A matrix pencil (A, B) is *regular* if $\det(A - \lambda B) \neq 0$ for at least one value $\lambda \in \mathbb{C}$ (equivalently, for almost every λ). We say (A, B) is *singular* if both A and B are singular and $A - \lambda B$ is singular for all $\lambda \in \mathbb{C}$.

We define the *generalized eigenvalues* of a matrix pencil:

DEFINITION 8. A *generalized eigenvalue* of a matrix pencil (A, B) is a number $\lambda_0 \in \mathbb{C}$ such that $A - \lambda_0 B$ is singular.

For a regular pencil (A, B), the *multiplicity* of a generalized eigenvalue λ_0 is the multiplicity of the root $\lambda = \lambda_0$ for the polynomial $\det(A - \lambda B)$ over λ.

If B is singular, we adopt the convention that ∞ is a generalized eigenvalue of the pencil (A, B) with multiplicity $n - \deg(\det(A - \lambda B))$.

Notably, every $\lambda \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ serves as a generalized eigenvalue of a singular matrix pencil.

Furthermore, the $SL(2, \mathbb{R})$-action for a pair (A, B) of $n \times n$ matrices induces changes in the generalized eigenvalues through a Möbius transformation:

LEMMA 1.2. *Let $\lambda_1, \ldots, \lambda_n$ denote the generalized eigenvalues of the matrix pencil (A, B). Then for any $p, q, r, s \in \mathbb{R}$ with $ps - qr \neq 0$, the generalized eigenvalues of $(pA + qB, rA + sB)$ are given by $\lambda_i' := \frac{p\lambda_i + q}{r\lambda_i + s}$, $i = 1, \ldots, n$.*

A matrix pencil (A, B) is *symmetric* if both A and B are symmetric matrices. We define *definiteness* for symmetric matrix pencils:

DEFINITION 9. A symmetric matrix pencil (A, B) is (semi-) definite, if either A or B is (semi-) definite, or if $A - \lambda B$ is (semi-) definite for at least one number $\lambda \in \mathbb{R}$.

We define congruence transformations of symmetric matrix pencils as

$$(A, B) \to (Q^{\mathrm{T}} A Q, Q^{\mathrm{T}} B Q),$$

where $Q \in GL(n, \mathbb{R})$, and $A, B \in Sym_n(\mathbb{R})$. It's worth noting that generalized eigenvalues remain invariant under these transformations.

Our work utilizes a normal form of matrix pencils under congruence transformation. We begin by introducing block-diagonal matrix pencils:

Definition 10. A *block-diagonal matrix pencil* is a matrix pencil (A, B), where $A = diag(A_1, \dots, A_m)$ and $B = diag(B_1, \dots, B_m)$; for $i = 1, \dots, m$, A_i and B_i are square matrices of the same dimension d_i.

The blocks of an $n \times n$ block-diagonal matrix pencil (A, B) define a partition of the set $\{1, \dots, n\}$. We say the matrix pencil (A', B') is (strictly) *finer* than the matrix pencil (A, B) if the partition corresponding to the pencil (A', B') is (strictly) finer than the one corresponding to (A, B), up to a permutation of numbers $1, \dots, n$.

Jordan canonical form characterizes the "finest" block-diagonalizations of regular symmetric matrix pencils:

Lemma 1.3 [**23**]. *Let (A, B) be a symmetric matrix pencil with B invertible. Suppose that the Jordan canonical form of $B^{-1}A$ is $Q^{-1}B^{-1}AQ = J = diag(J_1, \dots, J_m)$, where J_i is a Jordan block of dimension d_i, $i = 1, \dots, m$. Then $(A', B') = (Q^{\mathrm{T}}AQ, Q^{\mathrm{T}}BQ)$ is a block-diagonal matrix pencil; the block (A_i, B_i) is of dimension d_i for $i = 1, \dots, m$. Moreover, (A', B') is finer than any matrix pencil in its congruence equivalence class.*

Definition 11. For a regular symmetric matrix pencil (A, B), let c be any real number such that $B + cA$ is invertible, and $Q^{-1}(B + cA)^{-1}AQ$ is the Jordan canonical form of $(B + cA)^{-1}A$. Define the *normal form* of (A, B) under congruence transformations as

$$(A', B') = (Q^{\mathrm{T}}AQ, Q^{\mathrm{T}}BQ).$$

Symmetricity of A' and B' together with the fact that $A' = JB'$ implies the following, which further characterizes the diagonal blocks of the pencil (A', B'):

Lemma 1.4. *In the notation of Lemma 1.3, let (A_i, B_i) be the diagonal blocks of the congruence normal form (A', B') of the matrix pencil (A, B), $i = 1, \dots, m$. Suppose that $A_i = (a_i^{j,k})_{j,k=1}^{d_i}$ and $B_i = (b_i^{j,k})_{j,k=1}^{d_i}$. Then the entries $a_i^{j,k}$ satisfy:*

(i) $a_i^{j,k} = a_i^{j',k'}$, *for any $j + k = j' + k'$,*
(ii) $a_i^{j,k} = 0$, *for any $j + k \le d_i$.*

The entries $b_i^{j,k}$ satisfy the same property.

2. *Main Results*

Our first result focuses on step (2) in Poincaré's Algorithm. Following the sub-algorithm proposed in [**6**] for hyperbolic spaces, we adopt the *Blum-Shub-Smale (BSS) computational model*[**3**], where arbitrarily many real numbers can be stored, and rational functions over real numbers can be computed in a single step. However, this sub-algorithm cannot be fully extended to $\mathcal{P}(n)$ due to a fundamental distinction: while a hyperplane of $\mathbf{H}^n$ is isometric to $\mathbf{H}^{n-1}$, no analogous structure exists for $\mathcal{P}(n)$. To avoid this limitation, we introduce the following lemma:

LEMMA 2.1. *Let $B_1, \ldots, B_l \in Sym_n(\mathbb{R})$ be linearly independent matrices, and that $span(B_1, \ldots, B_l)$ contains an invertible element. Then $span(B_1, \ldots, B_l)$ contains a positive definite element if and only if*

$$\sum x_0^i B_i > 0 \tag{2.1}$$

holds for a real and isolated critical point $(x_0^1, \ldots, x_0^l)$ of the homogeneous polynomial $P(x^1, \ldots, x^l) = \det(\sum x^i B_i)$ restricted to the unit sphere $\mathbf{S}^{l-1}$.

Utilizing Lemma 2.1, we devise a sub-algorithm to address step (2) in the proposed Poincaré's Algorithm for $SL(n, \mathbb{R})$. Subsection 3.1 provides a detailed exposition of this sub-algorithm.

Our second result focuses on step (3) (ii) in Poincaré's Algorithm. We aim to establish a ridge cycle condition for convex polyhedra in $\mathcal{P}(n)$, analogously to similar conditions in hyperbolic spaces[**18**]. However, in $\mathcal{P}(n)$, the Riemannian angle should be substituted with an angle-like function satisfying specific natural properties [**11**]:

DEFINITION 12. An *invariant angle function* $\theta(-,-)$ is a function defined on a subset of the set of pairs of co-oriented hyperplanes (σ_1, σ_2) in $\mathcal{P}(n)$ with the following properties:

(i) For any co-oriented hyperplanes σ_1 and σ_2, $0 \le \theta(\sigma_1, \sigma_2) \le \pi$. Furthermore, $\theta(\sigma_1, \sigma_2) = 0$ if and only if $\sigma_1 = \sigma_2$, while $\theta(\sigma_1, \sigma_2) = \pi$ if and only if $\sigma_1 = -\sigma_2$.

(ii) For any co-oriented hyperplanes σ_1 and σ_2 and any $g \in SL(n, \mathbb{R})$, $\theta(g.\sigma_1, g.\sigma_2) = \theta(\sigma_1, \sigma_2)$.

(iii) For any co-oriented hyperplanes σ_1 and σ_2, $\theta(\sigma_2, \sigma_1) = \theta(\sigma_1, \sigma_2)$, $\theta(-\sigma_1, \sigma_2) = \pi - \theta(\sigma_1, \sigma_2)$.

(iv) For any co-oriented hyperplane σ_2 lying between σ_1 and σ_3, $\theta(\sigma_1, \sigma_2) + \theta(\sigma_2, \sigma_3) = \theta(\sigma_1, \sigma_3)$.

We proceed to formulate the ridge cycle condition:

DEFINITION 13. Let P be an exact convex polyhedron in $\mathcal{P}(n)$, with facet pairing Φ. Assume that θ is an invariant angle function defined on all pairs of hyperplanes of $\mathcal{P}(n)$ intersecting at a ridge of P. We say that P satisfies the *ridge cycle condition* if each ridge cycle $[x]$ of Φ satisfies the followings:

– The ridge cycle $[x]$ is a finite set, $[x] = \{x_1, \ldots, x_m\}$.
– The angle sum $\theta[x] = \sum_{i=1}^m \theta(x_i) = 2\pi/k$ for $k \in \mathbb{N}$. Here, $\theta(x_i)$ represents the invariant angle θ of the two co-oriented hyperplanes spanned by the two facets of P containing x_i.

We note that this ridge cycle condition does not depend on the choice of the invariant angle function θ. For any exact convex polyhedron P in $\mathcal{P}(n)$, the ridge cycle condition is equivalent to the tiling condition[**11**]. Indeed, let $g_i \in SL(n, \mathbb{R})$ be the facet pairing transformation that takes x_i to x_{i+1}, and U_i be a sufficiently small neighborhood of x_i in P for $i = 1, \ldots, n$. Then, the images $V_i = \left(\prod_{j=1}^{i-1} g_j\right)^{-1} .U_i$ and V_{i+1} share a facet. The ridge cycle condition for $[x]$ implies that the images $V_1, \ldots, V_n$ tile a neighborhood of x_1 in P, and vice versa.

We explicitly construct an invariant angle function for generic pairs of co-oriented hyperplanes, as presented in the main theorem below.

THEOREM 2.2. *Let linearly independent symmetric matrices A and B be normal vectors of co-oriented hyperplanes σ_1 and σ_2 in $\mathcal{P}(n)$, respectively. In addition, assume that the matrix* *pencil (A,B) is invertible. Denote the distinct generalized eigenvalues of (A,B) by $\lambda_1,\dots,\lambda_m$. Then:*

(i) *If there exists some nonreal numbers $\lambda_1,\dots,\lambda_k,\lambda_1^*,\dots,\lambda_k^*$ in the set of generalized eigenvalues of (A,B), the following is an invariant angle function:*

$$\gamma(\sigma_1,\sigma_2)=\frac{1}{k}\sum_{i=1}^{k}|\arg(\lambda_i)|\,. \tag{2.2}$$

(ii) *If all generalized eigenvalues of (A,B) are real (including ∞), ordered as $\lambda_k>\dots>\lambda_1$, and $k\geq 3$, the following is an invariant angle function (realized as a limit if $\lambda_k=\infty$ is a generalized eigenvalue):*

$$\gamma(\sigma_1,\sigma_2)=\arccos\frac{\sum_{i=1}^{k}\frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^{k}\frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^{k}\frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}, \tag{2.3}$$

where $\lambda_{k+1}=\lambda_1$.

(iii) *If all eigenvalues are real and $k\leq 2$, there is no invariant angle function defined on any non-empty domain containing the full orbit of (σ_1,σ_2) for the $SL(2,\mathbb{R})$-action.*

We present additional results concerning Dirichlet-Selberg domains and hyperplanes in $\mathcal{P}(n)$. Among these results, we aim to determine whether two hyperplanes of $SL(n,\mathbb{R})/SO(n)$ are disjoint. Based on a result due to Finsler, [**25**], we prove the following:

THEOREM 2.3. *Hyperplanes $A^\perp$ and $B^\perp$ in $\mathcal{P}(n)$ are disjoint if and only if either of the following holds, up to a congruence transformation of (A,B):*

(i) *The matrix pencil (A,B) is diagonal and semi-definite.*

(ii) *The matrix pencil (A,B) is block-diagonal, where the blocks are at most 2-dimensional. Moreover, all blocks (A_i,B_i) of dimension 2 share the same generalized eigenvalue λ, while $A-\lambda B$ is semi-definite.*

An algorithm detailing the procedure for determining the disjointness of hyperplanes is described in Subsection 3.3.

In addition, we establish a sufficient condition to ascertain if two Selberg bisectors $Bis(X,Y)$ and $Bis(Y,Z)$ are disjoint, analogously to the hyperbolic case in [**12**]. First we consider maximal flat totally geodesic submanifolds of $\mathcal{P}(n)$, which are isometric to the Euclidean $(n-1)$-space. In [**14**], one of these submanifolds is referred to as the *model flat* of $\mathcal{P}(n)$:

$$F_{mod}=\{diag(x_1,\dots,x_n)\,|\,x_i>0,\ \prod x_i=1\}.$$

Moreover, for any distinct points $X,Y\in\mathcal{P}(n)$, there is an isometry $g\in SL(n,\mathbb{R})$, such that $g.Y=I$ and $g.X\in F_{mod}$.

We divide the model flat into (2^n-2) chambers:

DEFINITION 14. The model flat F_{mod} of $\mathcal{P}(n)$ is partitioned into (2^n-2) chambers denoted by

$$\Delta^{\mathcal{I}}=\{X=\mathrm{diag}(x_i)\in F_{mod}|0<x_i<1,\ \forall i\in\mathcal{I};\quad x_i>1,\ \forall i\notin\mathcal{I}\}.$$

For any number $t\in(0,1)$, define

$$\Delta_t^{\mathcal{I}}=\left\{X\in\Delta^{\mathcal{I}}\,\middle|\,\frac{\min|\log x_i|}{\max|\log x_i|}\geq t\right\}.$$

$\Delta_t^{\mathcal{I}}$ is a cone contained in the chamber $\Delta^{\mathcal{I}}$ and is away from the chamber boundary.

The sufficient condition is presented in the theorem below:

THEOREM 2.4. *Let X, Y, Z be points in $\mathcal{P}(n)$, and $L = \min(s(Y, X), s(Y, Z))$. Let g_X and $g_Z \in SL(n, \mathbb{R})$ be elements such that*

$$g_X.Y = g_Z.Y = I, \; g_X.X \in F_{mod}, \; g_Z.Z \in F_{mod}.$$

Define θ as the maximum angle between the i-th column vector of $g_X^{-1}g_Z$ and the i-th standard unit vector for $i = 1, \ldots, n$.

Suppose that there exists $t \in (0, 1)$ and a subset $\mathcal{I} \subset \{1, \ldots, n\}$ such that the points $g_X.X \in \Delta_t^{\mathcal{I}}$, $g_Z.Z \in \Delta_t^{\mathcal{I}^c}$, and

$$\frac{1 + \sqrt{n-2}\sin\theta}{\cos\theta - \sqrt{n-2}\sin\theta} \leq \sqrt{t} \cdot \left(\frac{L-1}{n-1}\right)^{t/2}. \tag{2.4}$$

Then the bisectors $Bis(X, Y)$ and $Bis(Y, Z)$ in $\mathcal{P}(n)$ are disjoint.

We also investigate whether a subgroup of $SL(n, \mathbb{R})$ admits a finitely-sided Dirichlet-Selberg domain for a generic choice of center. This property, examined by Poincaré's Algorithm, implies the geometric finiteness, though the reverse is not always true. In particular, we categorize discrete Abelian subgroups of $SL(3, \mathbb{R})$ with exclusively positive eigenvalues based on whether their Dirichlet-Selberg domains are finitely-sided. We begin by exhausting all cases of such subgroups:

PROPOSITION 2.5. *Let Γ be a discrete Abelian subgroup of $SL(3, \mathbb{R})$ where all eigenvalues of each $\gamma \in \Gamma$ are positive real numbers. Then, Γ is conjugate to a subgroup of $SL(3, \mathbb{R})$ generated by either of the following:*

(i) For cyclic Γ, the generators are displayed below:

Type	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*
Generator	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t \end{pmatrix}$ *($r+s+t=0$; $r,s,t \neq 0$)*	$\begin{pmatrix} e^s & 0 & 0 \\ 0 & e^{-s} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *($s \neq 0$)*	$\begin{pmatrix} e^t & 1 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$ *($t \neq 0$)*

(ii) For 2-generated Γ, the generators are displayed below:

Type	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*
Generators	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t \end{pmatrix}$	$\begin{pmatrix} e^t & 1 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$
	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$ *($b \neq a(a-1)/2$)*	$\begin{pmatrix} e^{r'} & 0 & 0 \\ 0 & e^{s'} & 0 \\ 0 & 0 & e^{t'} \end{pmatrix}$ *($r+s+t = r'+s'+t' = 0$)*	$\begin{pmatrix} e^s & a & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-2s} \end{pmatrix}$ *($(s,t) \neq (0,0)$)*

The proof of Proposition 2.5 is elementary and is left to the reader. Our classification regarding the finite-sidedness of Dirichlet-Selberg domains is presented below.

THEOREM 2.6. *Let Γ be a discrete and free Abelian subgroup of $SL(3,\mathbb{R})$, generated by matrices with exclusively positive eigenvalues.*

- *If Γ is a cyclic group of type (1), (3), or (5), or if it is a 2-generated group of type (1) or (4), the Dirichlet-Selberg domain $DS(X,\Gamma)$ is finitely-sided for all $X \in \mathcal{P}(3)$.*
- *If Γ is a cyclic group of type (2) or (4), or if it is a 2-generated group of type (2), (3) or (5), the Dirichlet-Selberg domain $DS(X,\Gamma)$ is infinitely-sided for all X in a dense and Zariski open subset of $\mathcal{P}(3)$.*

Utilizing Dirichlet-Selberg domains in $\mathcal{P}(n)$, we extend the notion of *Schottky groups*[**17**] to subgroups of $SL(n,\mathbb{R})$:

DEFINITION 15. A discrete subgroup $\Gamma < SL(n,\mathbb{R})$ is called a *Schottky group* of rank k if there exists a point $X \in \mathcal{P}(n)$ such that the Dirichlet-Selberg domain $DS(X,\Gamma)$ is $2k$-sided and ridge-free.

Schottky groups in $SL(n,\mathbb{R})$ are free subgroups of $SL(n,\mathbb{R})$, analogously to the original notion in hyperbolic spaces. Our research investigates the existence of such groups among subgroups of $SL(n,\mathbb{R})$:

DEFINITION 16. For any $A \in SL(n,\mathbb{R})$ with only positive eigenvalues, one defines the *attracting and repulsing subspaces* of $\mathbb{R}^n$ as follows:

$$C_A^+ = span_{\lambda_i>1}(\mathbf{v}_i), \quad C_A^- = span_{0<\lambda_j<1}(\mathbf{v}_j),$$

where $\mathbf{v}_i$ denotes the eigenvector of A^{T} associated with the eigenvalue λ_i, $i = 1,\dots,n$.

THEOREM 2.7.

(i) *Suppose that n is even, and $A_1,\dots,A_k \in SL(n,\mathbb{R})$ are such that the attracting and repulsing spaces $C_{A_i}^{\pm}$, $i = 1,\dots,k$, are all $n/2$-dimensional and pairwise transversal. Then there exists an integer $M > 0$ such that the group $\Gamma = \langle A_1^M,\dots,A_k^M\rangle$ is a Schottky group of rank k.*

(ii) *Suppose that n is odd, and $A_1,\dots,A_k \in SL(n,\mathbb{R})$ generate a Schottky group and serve as the facet-pairing transformations. Then for at least one generator A_i, one of its eigenvalues λ satisfies that $|\lambda| = 1$.*

3. *Proof of the main results*

This section aims to prove the main results presented in Section 2 and demonstrate the connection of these results with the proposed Poincaré's Algorithm.

3.1. *Proof of Lemma 2.1, and description of step (2) in Poincaré's Algorithm*

We begin with the proof of Lemma 2.1:

Proof. The "if" part is self-evident. To prove the "only if" part, we assume that $X' = \sum x'^i B_i$ is a positive definite element in $span(B_1,\dots,B_l)$, where $(x'^i) := \mathbf{x}' \in \mathbb{R}^l$ is a unit vector. This is consistent with the lemma assumption.

We first show the existence of a critical point of $P|_{\mathbf{S}^{l-1}}$ satisfying (2.1). Let Σ be the connected component of $\mathbf{S}^{l-1}\backslash\{P(x^1,\dots,x^l)=0\}$ containing $\mathbf{x}'$. The region Σ contains a local maximum point $\mathbf{x_0}$ of $P|_{\mathbf{S}^{l-1}}$, which is the desired critical point.

We proceed to show that the critical point $\mathbf{x_0}$ is isolated. Suppose the opposite that $\mathbf{x_0}$ is contained in an algebraic variety S with dimension ≥ 1, consisting of critical points of $P|_{\mathbf{S}^{l-1}}$. By replacing $\mathbf{x_0}$ with another point in S, we assume that $\mathbf{x_0}$ is a regular point, contained in a smooth curve of critical points, with an expansion:

$$\mathbf{x}(t)=\mathbf{x_0}+t\mathbf{y_0}+t^2\mathbf{z_0}+O(t^3),\ \ |t|<\epsilon,$$

where $\epsilon>0$ and $\mathbf{y_0}\neq\mathbf{0}$.

Since the curve $\mathbf{x}$ lies in the unit sphere, both $\mathbf{y_0}$ and $\mathbf{z_0}+\frac{||\mathbf{y_0}||^2}{2}\mathbf{x_0}$ lie in $T_{\mathbf{x_0}}\mathbf{S}^{l-1}$. As $\mathbf{x_0}$ is a critical point, the vanishing of the derivative of P along these directions implies that

$$\operatorname{tr}(X_0^{-1}Y_0)=0,\quad \operatorname{tr}(X_0^{-1}Z_0)=-\frac{||\mathbf{y_0}||^2}{2}\operatorname{tr}(X_0^{-1}X_0)=-\frac{n}{2}||\mathbf{y_0}||^2,$$

where $X(t)=\sum x^i(t)B_i$, $X_0=\sum x_0^iB_i$, $Y_0=\sum y_0^iB_i$, and $Z_0=\sum z_0^iB_i$.

On the other hand, $\det(X(t))=P(\mathbf{x}(t))\equiv P(\mathbf{x_0})=\det(X_0)$, implying that:

$$\sum_{i=1}^n\lambda_i=0,\qquad \sum_{1\leq i<j\leq n}\lambda_i\lambda_j+\sum_{i=1}^n\mu_i=0,$$

where λ_i and μ_i, $i=1,\dots,n$, are the eigenvalues of $X_0^{-1}Y_0$ and $X_0^{-1}Z_0$, respectively, as real numbers. Combining the equations above, we obtain that

$$0\leq\sum_{i=1}^n\lambda_i^2=(\sum_{i=1}^n\lambda_i)^2-2(\sum_{i<j}\lambda_i\lambda_j)=2\sum_{i=1}^n\mu_i=2\operatorname{tr}(X_0^{-1}Z_0)=-n\sum_{i=1}^l||\mathbf{y_0}||^2<0,$$

which leads to a contradiction. □

Utilizing Lemma 2.1, we can describe the following algorithm in the BSS model:

COROLLARY 3.1. *There is a numerical algorithm with an input consisting of matrices $A_1,\dots,A_l\in Sym_n(\mathbb{R})$, yielding the following outcomes:*
- *If the intersection $\bigcap_{i=1}^l A_i^\perp=\varnothing$, the algorithm outputs* false.
- *If $\bigcap_{i=1}^l A_i^\perp$ is non-empty, the algorithm outputs* true *and provides a sample point in $\bigcap_{i=1}^l A_i^\perp$.*

Proof. Given the input $A_1,\dots,A_l\in Sym_n(\mathbb{R})$, we compute a basis of the orthogonal complement of $\operatorname{span}(A_1,\dots,A_l)$ in $Sym_n(\mathbb{R})$, denoted by $\{B_1,\dots,B_{l'}\}$. Then,

$$\bigcap_{i=1}^l A_i^\perp=span(B_1,\dots,B_{l'})\cap\mathcal{P}(n).$$

If $P(x^1,\dots,x^{l'})=\det(\sum x^iB_i)\equiv0$, then $\bigcap_{i=1}^l A_i^\perp$ is empty. Otherwise, $P(x^1,\dots,x^{l'})$ is a homogeneous polynomial of degree n in variables $x^1,\dots,x^{l'}$. The restriction of the polynomial $P(x^1,\dots,x^{l'})|_{\mathbf{S}^{l'-1}}$ has finitely many isolated critical points, found by numerical BSS algorithms, e.g., [**1**].

Let $\mathbf{x}_1,\dots,\mathbf{x}_m$ denote these isolated critical points, where $\mathbf{x}_j=(x_j^1,\dots,x_j^{l'})$. By Lemma 2.1, we determine if $\bigcap_{i=1}^{n-l} A_i^\perp$ is empty and generate a sample point of it by verifying if $\sum x_j^iB_i$ is positive definite for a certain $j\in\{1,\dots,m\}$. The algorithm we described terminates within a finite number of steps. □

Below, we describe step (ii) in Poincaré's Algorithm, utilizing Corollary 3.1 and referring to the algorithm for hyperbolic spaces described in [**6**].

Algorithm for Computing the Poset Structure of Polyhedra in $\mathcal{P}(n)$. Consider a point $X \in \mathcal{P}(n)$ and a list $\mathcal{A}'$ of matrices A'_i, where $i = 1, \ldots, k'$. Define the half-spaces $H_i = \{Y \in \mathcal{P}(n) | \mathrm{tr}(A'_i \cdot Y) \geq 0\}$ and the convex polyhedron $P_l = \bigcap_{i=1}^{l} H_i, l = 1, \ldots, k'$. We aim to describe an algorithm computing the face poset structure of $P_{k'}$ consisting of the following data:

- A subset $\mathcal{A} = \{A_1, \ldots, A_k\}$ of the input set $\mathcal{A}'$.
- A two-dimensional array L^{face} comprised of numbers from the set $\{1, \ldots, k'\}$, describing the set $\{F_1, \ldots, F_m\}$ of faces of $P_{k'}$. Specifically, L^{face} is a 2D array $\{L_1^{face}, \ldots, L_m^{face}\}$, where $m = |\mathcal{F}(P_{k'})|$, and such that
$$span(F_j) = \bigcap_{i \in L_j^{face}} A_i, \ j = 1, \ldots, m.$$
- A two-dimensional array L^{pos} comprised of numbers from the set $\{1, \ldots, m\}$, describing the inclusion relation among the faces of $P_{k'}$, namely
$$L_j^{pos} = \{1 \leq i \leq m | F_i \subsetneq F_j\}, \ j = 1, \ldots, m.$$
- An array L^{samp} of elements in $\mathcal{P}(n)$, serving to describe sample points associated with the faces of $P_{k'}$:
$$L_j^{samp} \in F_j, \ j = 1, \ldots, m.$$

Step (1). We aim to inductively obtain these data for each P_l, $l = 0, \ldots, k'$, and begin with $l = 0$. Since the polyhedron P_0 is the entire space $\mathcal{P}(n)$, we initialize
$$L^{face} = \{\varnothing\}, \ L^{pos} = \{\varnothing\}, \ L^{samp} = \{X\}, \text{ and } \mathcal{A} = \varnothing.$$

Step (2). We increase l by 1. Assume we have a set $\mathcal{A}$ of $n \times n$ symmetric matrices, such that $P_{l-1} = \bigcap_{A \in \mathcal{A}} \{\mathrm{tr}(A \cdot Y) \geq 0\}$, as well as lists by L^{face}, L^{pos}, and L^{samp} for P_{l-1} as described above. We describe the computation of these data of $P_l = P_{l-1} \cap H_l$ from which of P_{l-1}.

Step (3). To begin with, we remove the first element of the list $\mathcal{A}'$, denoted by A_l, and append it to $\mathcal{A}$.

Step (4). For any face $F \in \mathcal{F}(P_{l-1})$, exactly one of the following relative positions holds for the pair (F, H_l) [**6**]:

(1) The face $F \subset \partial H_l$.
(2) The face $F \subset \mathrm{int}(H_l)$.
(3) The face $F \subset H_l$, $F \cap \partial H_l \neq \varnothing$, and $F \cap \mathrm{int}(H_l) \neq \varnothing$.
(4) The face $F \cap H_l = \varnothing$.
(5) The face $F \cap \mathrm{int}(H_l) = \varnothing$, $F \cap \partial H_l \neq \varnothing$, and $F \cap H_l^c \neq \varnothing$.
(6) The face $F \cap \mathrm{int}(H_l) \neq \varnothing$ and $F \cap H_l^c \neq \varnothing$.

For $i = 1, \ldots, 6$, we denote $\mathcal{F}_{H_l}^{(i)}(P_{l-1})$ as the set of faces $F \in \mathcal{F}(P_{l-1})$ such that (F, H_l) belongs to relative position (i). We create a list L^{temp} of length $|\mathcal{F}(P_{l-1})|$ to represent the relative positions for these, initializing by $\{0, \ldots, 0\}$. We aim to replace the element L_j^{temp} with a number from $\{1, \ldots, 6\}$ indicating the relative position of (F_j, H_l), where $F_j \in \mathcal{F}(P_{l-1})$.

Step (5). We first determine the relative positions of the minimal faces in $\mathcal{F}(P_{l-1})$, i.e., the faces F_j such that $L_j^{pos} = \varnothing$, which are planes in $\mathcal{P}(n)$. These can be ascertained by checking if $F_j \cap H_l = \varnothing$ and computing $\mathrm{tr}(A_l X_j)$, where $X_j = L_j^{samp}$ is a sample point of F_j, with the sub-algorithm described in Corollary 3.1. We can thus determine whether (F_j, H_l) is in positions (1), (2), (4), or (6).

Step (6). We determine the relative position of non-minimal faces F_j, giving the relative positions of all proper faces of F_j. This is analogous to the corresponding step in [**6**].

Step (7). Utilizing the list L^{temp}, we derive data L^{face} for P_l, also analogously to [**6**]. Namely, the set of faces $\mathcal{F}(P_l)$ consists of:

– Faces $F \in \mathcal{F}^{(1)}_{H_l}(P_{l-1}) \cup \mathcal{F}^{(2)}_{H_l}(P_{l-1}) \cup \mathcal{F}^{(3)}_{H_l}(P_{l-1})$, and
– Intersections $F \cap H_l$ and $F \cap \partial H_l$, where $F \in \mathcal{F}^{(6)}_{H_l}(P_{l-1})$.

Step (8). We derive the data L^{pos} for P_l as follows. For $F \in \mathcal{F}^{(1)}_{H_l}(P_{l-1}) \cup \mathcal{F}^{(2)}_{H_l}(P_{l-1}) \cup \mathcal{F}^{(3)}_{H_l}(P_{l-1})$, the set of its proper faces in $\mathcal{F}(P_l)$ remains unchanged. For $F \in \mathcal{F}^{(6)}_{H_l}(P_{l-1})$, the proper faces of $F \cap H_l$ include:
– Proper faces F' of F, where $F' \in \mathcal{F}^{(1)}_{H_l}(P_{l-1}) \cup \mathcal{F}^{(2)}_{H_l}(P_{l-1}) \cup \mathcal{F}^{(3)}_{H_l}(P_{l-1})$,
– Intersections $F' \cap H_l$ and $F' \cap \partial H_l$, where $F' \in \mathcal{F}^{(6)}_{H_l}(P_{l-1})$ is a proper face of F, and
– The intersection $F \cap \partial H_l$.

For $F \in \mathcal{F}^{(6)}_{H_l}(P_{l-1})$, the proper faces of $F \cap \partial H_l$ include:
– Proper faces F' of F where $F' \in \mathcal{F}^{(1)}_{H_l}(P_{l-1})$, and
– Intersections $F' \cap \partial H_l$, where $F' \in \mathcal{F}^{(6)}_{H_l}(P_{l-1})$ is a proper face of F.

Step (9). We derive the data L^{samp} for P_l, i.e., the sample points of faces $F_j \cap H_l$ and $F_j \cap \partial H_l$ for $F_j \in \mathcal{F}^{(6)}_{H_l}(P_{l-1})$.

If $F_j \cap \partial H_l$ is a minimal face, its sample point is given by Corollary 3.1. If it has at least two proper faces, the sample point is given as the barycenter of the sample points of these proper faces, utilizing the data L^{pos}. If it has exactly one proper face, the sample point is given by perturbing the sample point of its proper face, similarly to [**6**].

The sample point of $F_j \cap H_l$ is derived by perturbing which of $F_j \cap \partial H_l$, analogously to [**6**].

Step (10). We check if all numbers in $\{1, \dots, l\}$ appear in L^{face} as facets. If a number $i \in \{1, \dots, l\}$ does not appear, we remove A_i from the list $\{A_1, \dots, A_l\}$, decrease by 1 any numbers greater than i appearing in L^{face}, and decrease l by 1.

Step (11). Repeat steps (2) through (10) if $\mathcal{A}'$ is non-empty. If $\mathcal{A}'$ is empty, the algorithm terminates, and the data $\mathcal{A}$, L^{face}, L^{pos} and L^{samp} are the required output of the algorithm.

3.2. *Proof of Theorem 2.2*

Recall the three types of regular matrix pencils mentioned in Theorem 2.2:
– Case (i): Some generalized eigenvalues of (A, B) are nonreal.
– Case (ii): The pencil (A, B) possesses at least three (distinct) generalized eigenvalues, all of which are real or infinity.
– Case (iii): The pencil (A, B) possesses at most two generalized eigenvalues, all real or infinity.

We say that a given pair of co-oriented hyperplanes $(A^\perp, B^\perp)$ in $\mathcal{P}(n)$ is of type (i), (ii), or (iii) if the pencil (A, B) corresponds to case (i), (ii), or (iii) above, respectively. By Lemma 1.2, it is evident that the hyperplane pairs (σ_1, σ_2) and $(g.\sigma_1, g.\sigma_2)$ share the same type for any $g \in SL(n, \mathbb{R})$. Furthermore, if σ_3 lies between σ_1 and σ_2, both (σ_1, σ_3) and (σ_2, σ_3) belong to the same type as (σ_1, σ_2). Consequently, we can independently prove the three statements in Theorem 2.2.

Case (i). We aim to prove that the function

$$\theta(\sigma_1, \sigma_2) = \frac{1}{k} \sum_{i=1}^{k} |\arg(\lambda_i)|$$

defined for all pairs (σ_1, σ_2) of type (i) satisfies the properties listed in Definition 12. Here, $\lambda_1, \dots, \lambda_k$ and their conjugates are the nonreal generalized eigenvalues of (A, B).

Proof of Theorem 2.2 (Case (i)). First, we establish the well-definedness of the function in equation (2.2). According to Lemma 1.2, the nonreal generalized eigenvalues of $(c_1 A, c_2 B)$ are $\frac{c_2}{c_1}\lambda_i$ and $\frac{c_2}{c_1}\lambda_i^*$, for any $c_1, c_2 > 0$. Since these have the same arguments as λ_i and λ_i^*, the value $\theta((c_1 A)^\perp, (c_2 B)^\perp) = \theta(A^\perp, B^\perp)$, i.e., the expression (2.2) remains unchanged. Furthermore, the arguments of λ_i and λ_i^* are opposite, ensuring that the expression (2.2) remains invariant when replacing λ_i with λ_i^*.

Next, we verify properties (i) to (iv) in Definition 12 for the function θ defined by (2.2). Property (i) is self-evident. For property (ii), we notice that the pencil $((g^{-1})^{\mathrm{T}}.A, (g^{-1})^{\mathrm{T}}.B)$ for hyperplanes $g.\sigma_1$ and $g.\sigma_2$ shares the same generalized eigenvalues as (A, B).

To verify property (iii), note that the pencil (B, A) possesses generalized eigenvalues λ_i^{-1} and λ_i^{-1*}, which have opposite arguments as λ_i and λ_i^*, respectively. Furthermore, the generalized eigenvalues of $(-A, B)$ are $-\lambda_i$ and $-\lambda_i^*$, while $|\arg(-\lambda_i)| = \pi - |\arg(\lambda_i)|$.

Lastly, we verify property (iv). Since positive rescalings of A and B preserve the values of $\theta(A^\perp, C^\perp)$, $\theta(C^\perp, A^\perp)$ and $\theta(A^\perp, B^\perp)$, we assume that $C = A + B$. Lemma 1.2 shows that the nonreal generalized eigenvalues of (A, C) are $(1 + \lambda_i)$ and $(1 + \lambda_i^*)$, while the nonreal generalized eigenvalues of (C, B) are $\frac{\lambda_i}{1+\lambda_i}$ and $\frac{\lambda_i^*}{1+\lambda_i^*}$, where $i = 1, \ldots, k$. Thus, property (iv) holds due to the product law of arguments.

Therefore, the function θ defined by (2.2) serves as an invariant angle function. $\square$

Case (ii). We aim to prove that the function

$$\theta(\sigma_1, \sigma_2) = \arccos \frac{\sum_{i=1}^k \frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}.$$

defined for all pairs (σ_1, σ_2) of type (ii) satisfies the properties listed in Definition 12. For clarity, we introduce the notation

$$t(x_1, \ldots, x_k) = \frac{\sum_{i=1}^k \frac{x_{i+1}+x_i}{x_{i+1}-x_i}}{\sqrt{\left(\sum_{i=1}^k \frac{1}{x_{i+1}-x_i}\right)\left(\sum_{i=1}^k \frac{(x_{i+1}+x_i)^2}{x_{i+1}-x_i}\right)}},$$

for any real numbers $x_1, \ldots, x_k$, and $t_>(x_1, \ldots, x_k) = t(x_{\sigma_k}, \ldots, x_{\sigma_1})$, where $\{\sigma_1, \ldots, \sigma_k\}$ represents the permutation of $\{1, \ldots, k\}$ such that $x_{\sigma_k} \geq \cdots \geq x_{\sigma_1}$.

We start with a lemma concerning the compositions of $t_>$ and Möbius transformations:

LEMMA 3.2. *Let φ be a Möbius transformation on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and let $\lambda_k > \cdots > \lambda_1$ represent real numbers. If φ is orientation-preserving, then*

$$t_>(\varphi(\lambda_1), \ldots, \varphi(\lambda_k)) = t(\varphi(\lambda_1), \ldots, \varphi(\lambda_k)). \tag{3.1}$$

If φ is orientation-reversing, then

$$t_>(\varphi(\lambda_1), \ldots, \varphi(\lambda_k)) = -t(\varphi(\lambda_1), \ldots, \varphi(\lambda_k)). \tag{3.2}$$

The proof of Lemma 3.2 is straightforward. We also require the following lemma:

LEMMA 3.3. *For any real $\lambda_k > \cdots > \lambda_1$,*

$$\sum_{i \neq j} \frac{(\lambda_{i+1} + \lambda_i - \lambda_{j+1} - \lambda_j)^2}{(\lambda_{i+1} - \lambda_i)(\lambda_{j+1} - \lambda_j)} > 0, \quad \sum_{i=1}^k \frac{(2 + \lambda_{i+1} + \lambda_i)^2}{\lambda_{i+1} - \lambda_i} > 0.$$

Lemma 3.3 is elementary.

We resume the proof of Theorem 2.2.

Proof of Theorem 2.2 (Case (ii)). Property (i) in Definition 12 demands us to show that (2.3) always yields values between 0 and π, i.e., $-1 < t(\lambda_1, \ldots, \lambda_k) < 1$ for any real numbers $\lambda_k > \cdots > \lambda_1$. Utilizing the Cauchy-Binet identity [**24**], we have:

$$\left(\sum_{i=1}^{k} \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^{k} \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right) - \left(\sum_{i=1}^{k} \frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}\right)^2 = \frac{1}{2}\sum_{i\neq j} \frac{(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}.$$

From Lemma 3.3, we deduce that the right-hand side is positive.

We proceed to prove the other properties. Property (ii) is proved similarly to the corresponding arguments in the preceding case.

For property (iii), note that the generalized eigenvalues of (B, A) are λ_i^{-1}, which result from an orientation-reversing Möbius transformation of λ_i, $i = 1, \ldots, k$. Lemma 3.2 implies that:

$$\cos\theta(\sigma_2, \sigma_1) = t_{>}(\lambda_1^{-1}, \ldots, \lambda_k^{-1}) = -t(\lambda_1^{-1}, \ldots, \lambda_k^{-1}),$$

and the summations in the expression of $t(\lambda_1^{-1}, \ldots, \lambda_k^{-1})$ simplify to:

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{1}{\lambda_i^{-1}-\lambda_{i+1}^{-1}} = \sum_{i=1}^{k}\left(\frac{1}{\lambda_i^{-1}-\lambda_{i+1}^{-1}} + \frac{\lambda_{i+1}-\lambda_i}{4}\right) = \sum_{i=1}^{k} \frac{(\lambda_{i+1}+\lambda_i)^2/4}{\lambda_{i+1}-\lambda_i},\\
&\sum_{i=1}^{k} \frac{(\lambda_i^{-1}+\lambda_{i+1}^{-1})^2}{\lambda_i^{-1}-\lambda_{i+1}^{-1}} = \sum_{i=1}^{k}\left(\frac{(\lambda_i^{-1}+\lambda_{i+1}^{-1})^2}{\lambda_i^{-1}-\lambda_{i+1}^{-1}} + (\lambda_{i+1}^{-1}-\lambda_i^{-1})\right) = \sum_{i=1}^{k} \frac{4}{\lambda_{i+1}-\lambda_i},\\
&\sum_{i=1}^{k} \frac{\lambda_i^{-1}+\lambda_{i+1}^{-1}}{\lambda_i^{-1}-\lambda_{i+1}^{-1}} = \sum_{i=1}^{k} \frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}.
\end{aligned}$$

This proves the former part of property (iii). For the latter part, note that the generalized eigenvalues of $(-A, B)$ are $-\lambda_1 > \cdots > -\lambda_k$.

Lastly, we address property (iv). Setting $\theta = \theta(\sigma_1, \sigma_2)$, $\theta_1 = \theta(\sigma_1, \sigma_3)$ and $\theta_2 = \theta(\sigma_3, \sigma_2)$, property (iv) reduces to

$$\cos(\theta) = \cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2). \tag{$*$}$$

Similarly to the the preceding case, we assume that $\sigma_3 = (A+B)^{\perp}$ without loss of generality. The generalized eigenvalues of $(A, A+B)$ are $(1+\lambda_i)$ and $(1+\lambda_i^*)$, and the generalized eigenvalues of $(A+B, B)$ are $\frac{\lambda_i}{1+\lambda_i}$ and $\frac{\lambda_i^*}{1+\lambda_i^*}$, $i = 1, \ldots, k$. Both sets are orientation-preserving Möbius transformations of λ_i and λ_i^*, $i = 1, \ldots, k$. Lemma 3.2 implies that

$$\cos(\theta_1) = \frac{\sum_{i=1}^{k} \frac{2+\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^{k} \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^{k} \frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}.$$

The summations in the expression of $\cos(\theta_2)$ are simplified to

$$\begin{aligned}
&\sum_{i=1}^{k}\left(\frac{\lambda_{i+1}+\lambda_i+2\lambda_i\lambda_{i+1}}{\lambda_{i+1}-\lambda_i} - \frac{\lambda_i-\lambda_{i+1}}{2}\right) = \frac{1}{2}\sum_{i=1}^{k} \frac{(2+\lambda_i+\lambda_{i+1})(\lambda_{i+1}+\lambda_i)}{\lambda_{i+1}-\lambda_i},\\
&\sum_{i=1}^{k}\left(\frac{(1+\lambda_i)(1+\lambda_{i+1})}{\lambda_{i+1}-\lambda_i} - \frac{\lambda_i-\lambda_{i+1}}{4}\right) = \frac{1}{4}\sum_{i=1}^{k} \frac{(2+\lambda_i+\lambda_{i+1})^2}{\lambda_{i+1}-\lambda_i},\\
&\sum_{i=1}^{k}\left(\frac{(\lambda_{i+1}+\lambda_i+2\lambda_i\lambda_{i+1})^2}{(1+\lambda_i)(1+\lambda_{i+1})(\lambda_{i+1}-\lambda_i)} - \frac{\lambda_i^2}{1+\lambda_i} + \frac{\lambda_{i+1}^2}{1+\lambda_{i+1}}\right) = \sum_{i=1}^{k} \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}.
\end{aligned}$$

Thus, Lemma 3.2 also implies that

$$\cos(\theta_2)=\frac{\sum_{i=1}^k\frac{(2+\lambda_i+\lambda_{i+1})(\lambda_{i+1}+\lambda_i)}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^k\frac{(2+\lambda_i+\lambda_{i+1})^2}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}.$$

By applying the Cauchy-Binet identity, we have:

$$\sin(\theta_1)=\frac{\sqrt{\left(\sum_{i=1}^k\frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)-\left(\sum_{i=1}^k\frac{2+\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}\right)^2}}{\sqrt{\left(\sum_{i=1}^k\frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}$$
$$=\frac{\sqrt{\frac{1}{2}\sum_{i\neq j}\frac{(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}}}{\sqrt{\left(\sum_{i=1}^k\frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}},$$

and

$$\sin(\theta_2)=\frac{\sqrt{\left(\sum_{i=1}^k\frac{(2+\lambda_i+\lambda_{i+1})^2}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)-\left(\sum_{i=1}^k\frac{(2+\lambda_i+\lambda_{i+1})(\lambda_{i+1}+\lambda_i)}{\lambda_{i+1}-\lambda_i}\right)^2}}{\sqrt{\left(\sum_{i=1}^k\frac{(2+\lambda_i+\lambda_{i+1})^2}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}$$
$$=\frac{\sqrt{\frac{1}{2}\sum_{i\neq j}\frac{4(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}}}{\sqrt{\left(\sum_{i=1}^k\frac{(2+\lambda_i+\lambda_{i+1})^2}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}.$$

Inequalities in Lemma 3.3 imply that

$$\sin(\theta_1)\sin(\theta_2)=\frac{\frac{1}{2}\sum_{i\neq j}\frac{2(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}}{\left(\sum_{i=1}^k\frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)\sqrt{\left(\sum_{i=1}^k\frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}.$$

By combining the equations above and using the Cauchy-Binet identity again, we have

$$\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)$$
$$=\frac{\left(\sum_{i=1}^k\frac{2+\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(2+\lambda_i+\lambda_{i+1})(\lambda_{i+1}+\lambda_i)}{\lambda_{i+1}-\lambda_i}\right)-\frac{1}{2}\sum_{i\neq j}\frac{2(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}}{\left(\sum_{i=1}^k\frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)\sqrt{\left(\sum_{i=1}^k\frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}$$
$$=\frac{\left(\sum_{i=1}^n\frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^n\frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}{\left(\sum_{i=1}^k\frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)\sqrt{\left(\sum_{i=1}^k\frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k\frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}=\cos(\theta).$$

This proves property (iv) in Definition 12. Therefore, the function θ given by (2.3) is an invariant angle function. □

Case (iii). To prove statement (3) in Theorem 2.2, we begin by establishing the following lemma:

LEMMA 3.4. *Let $K_l = \sum_{s+t=r+l} \mathbf{e}_s \otimes \mathbf{e}_t$, $l = 1, \dots, r$, be $r \times r$ matrices. Define that*

$$X = \sum_{l=1}^{r} x_l K_l, \quad \bar{X} = \sum_{l=1}^{r-1} x_l K_{l+1}.$$

Then for any $s > 0$ and $t \in \mathbb{R}$, there exists an element $g \in GL^+(r, \mathbb{R})$ satisfying the conditions:

$$g.\bar{X} = \bar{X}, \tag{3.3}$$

$$g.X = sX + t\bar{X}. \tag{3.4}$$

Proof. We claim the existence of a matrix g of the form

$$g = \sum_{l \le j} s^{r/2-j+1} p_l^{(j-l)} \mathbf{e}_l \otimes \mathbf{e}_j \tag{3.5}$$

that satisfies (3.3) and (3.4).

If g follows equation (3.5), the entries above the anti-diagonal of both equations and those on the anti-diagonal of (3.3) vanish. We will prove by induction on k that there exist numbers $p_l^{(k)} \in \mathbb{R}$, where $l = 1, \dots, r-k$, such that all entries under the anti-diagonal of both equations and those on the anti-diagonal of (3.4) equal on both sides.

We start with the base case $k = 0$. If we set $p_l^{(0)} = 1$ for $l = 1, \dots, r$, then the $(l+1, r+2-l)$ entries of both sides of (3.3) are equal to x_1, and the $(l+1, r+1-l)$ entries of both sides of (3.4) are equal to sx_1, where $l = 1, \dots, r-1$. Entries above these depend on $p_l^{(1)}$ and do not need to be discussed here.

We proceed to the general case $k > 0$, assuming that the solutions $p_l^{(k')}$ are determined for $0 \le k' < k$. The $(l+k, r+2-l)$ entries of (3.3), where $l = 2, \dots, r-k$, yield $(r-k-1)$ equations in unknowns $p_1^{(k)}, \dots, p_{r-k}^{(k)}$; the symmetricity of $g.\bar{X}$ reduces the number of equations to $\lfloor \frac{r-k}{2} \rfloor$. The $(l+k, r+1-l)$ entries of (3.4), where $l = 1, \dots, r-k$, yield $(r-k)$ equations in unknowns $p_1^{(k)}, \dots, p_{r-k}^{(k)}$, and the symmetricity of $g.X$ reduces the number of equations to $\lfloor \frac{r-k+1}{2} \rfloor$. Combining these equations yields a system of $(r-k)$ linear equations, which is upper-triangular if the $(r-k)$ unknowns are arranged as $p_1^{(k)}, p_{r-k}^{(k)}, p_2^{(k)}, \dots, p_{\lfloor \frac{r-k}{2}+1 \rfloor}^{(k)}$. Thus, a unique solution $p_1^{(k)}, \dots, p_{r-k}^{(k)}$ exists, dependent on s, t, $x_1, \dots, x_{k+1}$ and $p_j^{(k')}$, where $1 \le j \le r-k'$ and $k' < k$.

By induction, a solution set $p_l^{(k)}$ exists in terms of $x_1, \dots, x_r$, s, and t, where $k = 1, \dots, r-1$ and $l = 1, \dots, r-k$. Thus, there exists a matrix g satisfying (3.3) and (3.4). $\square$

Lemma 3.4 implies the following:

LEMMA 3.5. *(1) Suppose that (A, B) is a regular pencil of symmetric $n \times n$ matrices with only one distinct eigenvalue $\lambda \in \mathbb{R}$, and let $C = A - \lambda B$. Then, for any $s > 0$ and $t \in \mathbb{R}$, there exists an element $g \in GL^+(n, \mathbb{R})$ such that:*

$$g.C = C, \quad g.B = sB + tC.$$

(2) Suppose that (A, B) is a regular pencil of symmetric $n \times n$ matrices with only two distinct eigenvalues $\lambda, \lambda' \in \mathbb{R}$, and let $C = A - \lambda B$, $C' = A - \lambda' B$. Then for any $s, s' > 0$, there exists an element $g \in GL^+(n, \mathbb{R})$ such that:

$$g.C = sC, \quad g.C' = s'C'.$$

Proof. (1) Suppose that the pencil (A, B) has one distinct eigenvalue $\lambda \in \mathbb{R}$. According to Lemma 1.3, we may assume that the matrix pencil is in the normal form:

$$A = \operatorname{diag}(A_1, \dots, A_k),$$

and

$$B = \operatorname{diag}(B_1, \dots, B_k),$$

where $A_j = B_j J_{\lambda,r_j}$, and J_{λ,r_j} denotes the Jordan block of dimension r_j and eigenvalue λ.

Let $\tilde{B}_j = B_j - \lambda A_j = B_j J_{0,r_j}$. According to Lemma 3.4, for any $s > 0$ and $t \in \mathbb{R}$, there exist elements $g_j \in GL^+(r_j, \mathbb{R})$ such that:

$$g_j.\tilde{B}_j = \tilde{B}_j, \quad g_j.B_j = sB_j + t\tilde{B}_j.$$

Let $g = \operatorname{diag}(g_1, \dots, g_k) \in GL^+(n, \mathbb{R})$, then $g.C = C$ and $g.B = sB + tC$.

(2) Suppose that the pencil (A, B) has exactly two distinct eigenvalues $\lambda, \lambda' \in \mathbb{R}$. We may assume that the matrix pencil is in the normal form:

$$A = \operatorname{diag}(A_1, \dots, A_k, A'_1, \dots, A'_l),$$

and

$$B = \operatorname{diag}(B_1, \dots, B_k, B'_1, \dots, B'_l),$$

where $A_j = B_j J_{\lambda,r_j}$ and $A'_j = B'_j J_{\lambda',r'_j}$. Let $\tilde{B}_j = B_j J_{0,r_j}$ and $\tilde{B}'_j = B'_j J_{0,r'_j}$. According to Lemma 3.4, for any $s, t > 0$, there exist matrices g_j, $j = 1, \dots, k$, such that

$$g_j.\tilde{B}_j = \tilde{B}_j, \quad g_j.B_j = \frac{t}{s}B_j + \frac{t-s}{s(\lambda - \lambda')}\tilde{B}_j,$$

and matrices g'_j, $j = 1, \dots, l$, such that

$$g'_j.\tilde{B}'_j = \tilde{B}'_j, \quad g_j.B'_j = \frac{s}{t}B'_j + \frac{s-t}{t(\lambda' - \lambda)}\tilde{B}'_j.$$

Let $g = \operatorname{diag}(\sqrt{s}g_1, \dots, \sqrt{s}g_k, \sqrt{t}g'_1, \dots, \sqrt{t}g'_l) \in GL^+(n, \mathbb{R})$, then $g.C = sC$ and $g.C' = tC'$. □

We now return to part (iii) of Theorem 2.2.

Proof of Theorem 2.2 (Case (iii)). A positive rescaling of the normal vector does not change the associated co-oriented hyperplane. Therefore, we may replace the part "$g \in SL(n, \mathbb{R})$" in Definition 12 with "$g \in GL^+(n, \mathbb{R})$".

(1) Suppose that (A, B) has only one eigenvalue $\lambda \in \mathbb{R}$ and $C = A - \lambda B$. By replacing (A, B) with an element in its $SL(2, \mathbb{R})$-orbit, we may assume $\lambda > 0$. By Lemma 3.5, there is an element $g \in GL^+(n, \mathbb{R})$, such that $g.C = C$ and $g.B = \lambda B + C = A$. Denote $\sigma_k = (g^{1-k}.A)^\perp$, which is compatible with the notations $\sigma_1 = A^\perp$ and $\sigma_2 = B^\perp$. Then,

$$\theta(\sigma_k, \sigma_{k+1}) = \theta((g^{1-k}.A)^\perp, (g^{1-k}.B)^\perp) = \theta(A^\perp, B^\perp) = \theta(\sigma_1, \sigma_2).$$

One verifies that

$$(\lambda^k - 1)g^{1-k}.A = (\lambda^k - \lambda)g^{-k}.A + (\lambda - 1)A,$$

i.e., θ_k lies between θ_1 and θ_{k+1}. Property (iv) of invariant angle functions implies that

$$\theta(\sigma_1, \sigma_m) = \sum_{k=1}^{m-1} \theta(\sigma_k, \sigma_{k+1}) = (m-1)\theta(\sigma_1, \sigma_2)$$

for any $m > 1$, greater than π for m sufficiently large, a contradiction.

(2) Suppose that (A, B) has two eigenvalues λ and λ', $C = A - \lambda B$, and $C' = A - \lambda' B$. By replacing (A, B) with an element in its $SL(2, \mathbb{R})$-orbit, we may assume $\lambda > \lambda' > 0$. Lemma 3.5

implies the existence of an element $g \in GL^+(n, \mathbb{R})$ such that $g.A = B$, $g.C$ differs from C by a positive multiple, and $g.C'$ differs from C' by a positive multiple. Denote $\sigma_k = (g^{1-k}.A)^\perp$, $k \in \mathbb{N}$; we verify that σ_k lies between σ_1 and σ_{k+1} for all $k > 1$. Similarly to part (1), the value of $\theta(\sigma_1, \sigma_m)$ exceeds π for m large enough, leading to a contradiction.

In conclusion, for all symmetric matrix pencils (A, B) of type (iii), there are no invariant angle functions defined on its entire $SL(2, \mathbb{R})$-orbit. □

3.3. *Proof of Theorem 2.3, and the algorithm determining disjoint hyperplanes*

We will prove Theorem 2.3 in this section, which describes an equivalent condition for that two hyperplanes in $\mathcal{P}(n)$ are disjoint. We begin by reviewing a result of Finsler [**25**], providing an equivalent condition for $\bigcap_{i\in\mathcal{I}} \sigma_i \neq \varnothing$, where $\mathcal{I}$ is a finite set.

We introduce Σ as the collection of the hyperplanes σ_i for $i \in \mathcal{I}$. Moreover, $\mathcal{A}$ denotes the collection of the corresponding normal vectors A_i, which are symmetric matrices. We define the definiteness of a collection of symmetric $n \times n$ matrices:

Definition 17. We say the collection $\mathcal{A} = \{A_i \in Sym_n(\mathbb{R}) | i \in \mathcal{I}\}$ is (semi-) definite if there exist numbers $c_i \in \mathbb{R}$ for $i \in \mathcal{I}$ such that

$$A = \sum_{i\in\mathcal{I}} c_i A_i$$

is a non-zero positive (semi-) definite matrix.

Further, we introduce notation related to the Satake compactification $\overline{\mathcal{P}(n)} \subset \mathbf{P}(Sym_n(\mathbb{R}))$:

Definition 18. For $A \in Sym_n(\mathbb{R})$, define

$$N(A) = \{X \in \mathbf{P}(Sym_n(\mathbb{R})) | \operatorname{tr}(A \cdot X) = 0\},$$

and define $\overline{A^\perp} = \overline{\mathcal{P}(n)} \cap N(A)$.

The relationship between the definiteness of $\mathcal{A} = A_i$ and the emptiness of the intersection $\bigcap A_i^\perp$ is described by the following lemma:

Lemma 3.6 (cf. [**25**]). *The collection $\mathcal{A} = \{A_i\}_{i=1}^k$ of $n \times n$ symmetric matrices is semi-definite if and only if the intersection $\bigcap_{i=1}^k A_i^\perp$ is empty. Furthermore, $\mathcal{A}$ is (strictly) definite if and only if $\bigcap \overline{A_i^\perp} = \varnothing$.*

Proof. The proof for the case of $k = 1$ is straightforward, achieved by applying an $SL(n, \mathbb{R})$-action and assuming that A is diagonal. To extend the proof to general $k \in \mathbb{N}$, one notices the following: if $\bigcap \overline{A_i^\perp}$ is empty, then the subspace $\bigcap N(A_i) \subset \mathbf{P}(Sym(n))$ is disjoint from the closed convex region $\overline{\mathcal{P}(n)} \subset \mathbf{P}(Sym(n))$. Therefore, there exists a support hyperplane $N(B) \subset \mathbf{P}(Sym(n))$ such that $\bigcap N(A_i) \subseteq N(B)$ and $N(B) \cap \overline{\mathcal{P}(n)} = \varnothing$. □

To continue proving Theorem 2.3, we first examine the case where (A, B) is a regular pencil.

Case (1). Assume that (A, B) constitutes a regular pencil. If two hyperplanes $A^\perp$ and $B^\perp$ are disjoint, we have the following supplement to Lemma 3.6:

LEMMA 3.7. *If two hyperplanes $A^\perp$ and $B^\perp$ in $\mathcal{P}(n)$ are disjoint and (A, B) is regular, then all generalized eigenvalues of (A, B) are real numbers.*

The proof of Lemma 3.7 relies on certain algebraic results:

LEMMA 3.8. *Let t_0 be a real generalized eigenvalue of a symmetric $n \times n$ matrix pencil (A, B). We define a continuous function $\lambda(t)$ in a neighborhood of $t = t_0$ such that $\lambda(t)$ is an eigenvalue of $A - Bt$ and $\lambda(t_0) = 0$.*

Then, in a neighborhood of $t = t_0$, the function $\lambda(t)$ can be expressed as a product:

$$\lambda(t) = (t - t_0)^s \varphi(t),$$

where $s \in \mathbb{N}_+$ and $\varphi(t)$ is a continuous function with $\varphi(t_0) \neq 0$.

Proof. Around $(\lambda, t) = (0, t_0)$, $\lambda(t)$ has a Puiseux series expansion (see, e.g., [**2**]) with fractional exponents of denominator d. If $d \geq 2$, then some of the eigenvalues are not real in a punctured neighborhood of $t = t_0$, leading to a contradiction. Hence $d = 1$, and the conclusion follows. □

Proof of Lemma 3.7. First, we assume that $\overline{A^\perp}$ and $\overline{B^\perp}$ are disjoint, indicating that the pencil (A, B) is (strictly) definite. By applying an $SL(2, \mathbb{R})$-action on (A, B), we assume that B is positive definite, without altering the conclusion as per Lemma 1.2. Suppose that the polynomial $\det(A - tB)$ has distinct real zeroes t_i of multiplicity r_i, where $i = 1, \ldots, k$. For each i, there is a neighborhood $U_i \supset t_i$, on which the eigenvalues of $(A - tB)$ are smooth functions $\lambda_j(t)$ of t, $j = 1, \ldots, n$.

Suppose the signature of $A - \lambda_i B$ changes by $2k$ at $t = t_i$, i.e., k eigenvalues among $\lambda_j(t)$, $j = 1, \ldots, n$, change the sign at $t = t_i$. Lemma 3.8 implies that the determinant

$$\det(A - tB) = \prod_{j=1}^{n} \lambda_j(t),$$

a polynomial in t, has a factor $(t - t_i)^k$. That is, the zero t_i of $\det(A - tB)$ is of multiplicity at least k. Since B is positive definite, the signature of $(A - tB)$ increases by $2n$ from $-M$ to M for a sufficient large $M < \infty$, implying that $(A - tB)$ has at least n real zeroes between $-M$ and M (counting multiplicity). Consequently, all generalized eigenvalues of (A, B) are real.

Next, we assume that $A^\perp$ and $B^\perp$ are disjoint, indicating that (A, B) is semi-definite. We demonstrate that that all generalized eigenvalues of (A, B) are real by considering a sequence $\{(A_i, B_i)\}_{i=1}^\infty$ of strictly definite matrix pencils approximating (A, B). □

We proceed with the proof of Theorem 2.3:

Proof of Theorem 2.3 (regular case). The "if" part is straightforward. For the "only if" part, we assume that B is invertible. Furthermore, we consider that (A, B) is a real block-diagonal matrix pencil, $B^{-1}A$ is a real matrix in Jordan normal form, with Jordan blocks of the same dimensions as the block-diagonal pencil (A, B).

Suppose that (A, B) contains a block (A_i, B_i) of dimension $d_i \geq 3$. Utilizing Lemma 1.4, it's evident that all elements in (A_i, B_i) are indefinite, implying that (A_i, B_i) is is an indefinite pencil, a contradiction.

Now suppose that (A, B) contains a block (A_j, B_j) of dimension 2. Similar reasoning via Lemma 1.4 suggests that $A_j - \lambda B_j$ is the only possible semi-definite element, where λ is the generalized eigenvalue of (A_j, B_j). Consequently, all blocks of dimension 2 share the same eigenvalue λ.

Lastly, if (A, B) is diagonal, hyperplanes $A^\perp$ and $B^\perp$ are disjoint if and only if (A, B) is semi-definite. □

Case (2). Consider now the case where (A, B) is a singular pencil. If the pencil (A, B) arises from a lower-dimensional pencil, we have the following lemma:

LEMMA 3.9. *Suppose that $A_0, B_0 \in Sym_m(\mathbb{R})$ and $A = \mathrm{diag}(A_0, O), B = \mathrm{diag}(B_0, O) \in Sym_n(\mathbb{R})$. Then $A^\perp \cap B^\perp = \varnothing$ if and only if $A_0^\perp \cap B_0^\perp = \varnothing$ (in $\mathcal{P}(m)$).*

The proof of the Lemma is evident.

Additionally, we will utilize the following result:

LEMMA 3.10 [**10**]. *Let (A, B) be a singular symmetric $n \times n$ matrix pencil. Then (A, B) is congruent to (A', B'), where the matrices A' and B' satisfy*

$$A' = \begin{pmatrix} A_1 & O & O \\ O & O & O \\ O & O & O \end{pmatrix}, \quad B' = \begin{pmatrix} B_1 & B_2 & O \\ B_2^{\mathrm{T}} & O & O \\ O & O & B_3 \end{pmatrix},$$

for $n_1 \times n_1$ matrices A_1 and B_1, an $n_1 \times n_2$ matrix B_2, and an $n_3 \times n_3$ matrix B_3, where $n_1 + n_2 + n_3 = n$. Moreover, A_1 and B_3 are invertible.

We proceed with the proof of Theorem 2.3:

Proof of Theorem 2.3 (singular case). The "if" part is clear. For the "only if" part, from Lemma 3.10, we observe that (A, B) is congruent to both:

$$P^{\mathrm{T}}AP = \begin{pmatrix} A_1 & O & O \\ O & O & O \\ O & O & O \end{pmatrix}, \quad P^{\mathrm{T}}BP = \begin{pmatrix} B_1 & B_2 & O \\ B_2^{\mathrm{T}} & O & O \\ O & O & B_3 \end{pmatrix}, \tag{3.6}$$

and

$$P'^{\mathrm{T}}AP' = \begin{pmatrix} A_1' & A_2' & O \\ A_2'^{\mathrm{T}} & O & O \\ O & O & A_3 \end{pmatrix}, \quad P'^{\mathrm{T}}BP' = \begin{pmatrix} B_1' & O & O \\ O & O & O \\ O & O & O \end{pmatrix}, \tag{3.7}$$

where A_1, B_3, A_3' and B_1' are invertible.

If both A_2' and B_2 are nonzero, we construct a positive definite matrix orthogonal to A and B as follows. The nonzero A_2' implies that A is indefinite, so is A_1. According to Lemma 3.6, there is a positive definite matrix X_1 perpendicular to A_1. As $B_2 \neq O$, there is a matrix X_2 such that

$$2\mathrm{tr}(X_2 \cdot B_2^{\mathrm{T}}) + \mathrm{tr}(X_1 \cdot B_1) + \mathrm{tr}(B_3) = 0.$$

Since X_1 is positive definite, there exists a positive definite matrix X_4, such that

$$\begin{pmatrix} X_1 & X_2 \\ X_2^{\mathrm{T}} & X_4 \end{pmatrix}$$

is positive definite. Hence,

$$X := P \cdot \begin{pmatrix} X_1 & X_2 & O \\ X_2^{\mathrm{T}} & X_4 & O \\ O & O & I \end{pmatrix} \cdot P^{\mathrm{T}} \in A^{\perp} \cap B^{\perp}.$$

Therefore, $A^{\perp}$ and $B^{\perp}$ are disjoint only if either $A_2' = O$ or $B_2 = O$. Without loss of generality, suppose that $B_2 = O$, then (A, B) is congruent to $(\mathrm{diag}(A_0, O_{n-m}), \mathrm{diag}(B_0, O_{n-m}))$, where $(A_0, B_0) := (\mathrm{diag}(A_1, O), \mathrm{diag}(B_1, B_3))$ is an invertible pencil of dimension m. Applying Theorem 2.3 (regular case) to (A_0, B_0), we conclude that (A, B) satisfies either condition (i) or (ii). □

Derived from Theorem 2.3, we describe an algorithm that checks if two hyperplanes $A^{\perp}$ and $B^{\perp}$ are disjoint:

Algorithm for certifying disjointness of two hyperplanes. For given normal vectors $A, B \in Sym_n(\mathbb{R})$ of hyperplanes in $\mathcal{P}(n)$, we follow these steps to ascertain if $A^{\perp} \cap B^{\perp} = \varnothing$:

(1) Determine if (A, B) is regular by computing the coefficients of $\det(A - tB)$.
(2) If (A, B) is regular, assume that A is invertible without loss of generality. Compute the Jordan normal form of $A^{-1}B = PJP^{-1}$ using the standard algorithm.
(3) If any Jordan block of J has dimension ≥ 3, then $A^{\perp}$ and $B^{\perp}$ are not disjoint.
(4) Otherwise, compute $A_0 = P^{\mathrm{T}}AP$ and $B_0 = P^{\mathrm{T}}BP$. If J has blocks of dimension 2, check if all these blocks share the same eigenvalue λ and if the diagonal matrix $A_0 - \lambda B_0$ is semi-definite. This condition holds if and only if $A^{\perp} \cap B^{\perp} = \varnothing$.
(5) If J is diagonal, check if the diagonal matrices A_0 and B_0 have a positive semi-definite linear combination. This condition holds if and only if $A^{\perp} \cap B^{\perp} = \varnothing$.
(6) If (A, B) is singular, compute the standard form of (A, B) as in equations (3.6) and (3.7) following the algorithm described in [**10**].
(7) In the standard form mentioned above, if both matrices B_2 and A_2' are nonzero, then $A^{\perp}$ and $B^{\perp}$ are not disjoint.
(8) Otherwise, assume that $B_2 = O$, let $A_0 = \mathrm{diag}(A_1, O)$ and $B_0 = \mathrm{diag}(B_1, B_3)$. Check if $A_0^{\perp} \cap B_0^{\perp} = \varnothing$ by performing steps (2) to (5). This is equivalent to that $A^{\perp} \cap B^{\perp} = \varnothing$.

3.4. *Proof of Theorem 2.4*

We prove Theorem 2.4 after a few lemmas:

LEMMA 3.11. *Let $X = \mathrm{diag}(x_i) \in \Delta_t^{\mathcal{I}}$ and $s(I, X) \geq L$, $L \geq n$. Then for any $i \in \mathcal{I}$ and $j \in \mathcal{I}^{\mathsf{c}}$,*

$$\frac{|x_i^{-1} - 1|}{|x_j^{-1} - 1|} \geq t \cdot \left(\frac{L-1}{n-1}\right)^t.$$

Proof. The Lemma's assumption implies the existence of $u > 0$ such that $x_i \in [e^{-u}, e^{-tu}]$ for $i \in \mathcal{I}$, and $x_i \in [e^{tu}, e^u]$ for $i \in \mathcal{I}^{\mathsf{c}}$. Suppose that $|\mathcal{I}| = k$, $1 \leq k \leq n-1$. We deduce that

$$L \leq \sum x_i \leq ke^{-tu} + (n-k)e^u \leq k + (n-k)e^u,$$

thus $e^u \geq (L-k)/(n-k) \geq (L-1)/(n-1)$. It follows that

$$\frac{|x_i^{-1} - 1|}{|x_j^{-1} - 1|} \geq \frac{e^{tu} - 1}{1 - e^{-u}} \geq te^{tu} \geq t \cdot \left(\frac{L-1}{n-1}\right)^t,$$

for any $i \in \mathcal{I}$ and $j \in \mathcal{I}^{\mathsf{c}}$. □

The two lemmas below are self-evident:

Lemma 3.12. *Suppose that* $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \in SO(n)$, *where* $g_1 \in Mat_k(\mathbb{R})$. *Then,* $g = g_+ g_-^{-1}$, *where*

$$g_+ = \begin{pmatrix} (g_1^{-1})^{\mathrm{T}} & -(g_1^{-1})^{\mathrm{T}} g_3^{\mathrm{T}} \\ O & I \end{pmatrix}, \quad g_- = \begin{pmatrix} I & O \\ -g_4^{-1} g_3 & g_4^{-1} \end{pmatrix}.$$

Lemma 3.13. *Define*

$$\sigma_r(A) = \max_i \sum_{j=1}^n |a_{ij}|, \quad \sigma_c(A) = \max_j \sum_{i=1}^n |a_{ij}|$$

for a matrix $A = (a_{ij}) \in Mat_n(\mathbb{R})$. *If there exist elements* $A, B \in Mat_n(\mathbb{R})$ *such that* $\sigma_r(A) \leq a$ *and* $\sigma_r(B) \leq b$, *then* $\sigma_r(AB) \leq ab$. *A similar conclusion holds for* σ_c.

Utilizing Lemma 3.13, we have the following result:

Lemma 3.14. *Consider a matrix* $A = (a_{ij}) \in Mat_n(\mathbb{R})$, *where* $a_{ii} \geq a$ *and* $\sum_{j \neq i} |a_{ij}| \leq a'$ *for all* $i = 1, \dots, n$, *and* $a > a'$ *are real numbers. Then* A *is invertible, with* $\sigma_r(A^{-1}) \leq \frac{1}{a-a'}$. *A similar conclusion holds for* σ_c.

Proof. This follows directly from Lemma 3.13, with noticing that

$$A^{-1} = A_1 A_2^{-1} = A_1 \sum_{k=0}^{\infty} (I - A_2)^k,$$

where

$$A_1 = \mathrm{diag}(a_{ii}^{-1}), \ A_2 = (a_{ij}/a_{ii})_{i,j=1}^n.$$

□

We turn to the proof of Theorem 2.4.

Proof of Theorem 2.4. By applying the $SL(n,\mathbb{R})$-action, we can assume that $Y = I$, X is diagonal, and $\mathcal{I} = \{k+1, \dots, n\}$, where $1 \leq k < n$. Let $g = g_Z = (g_{ij}) \in SO(n)$, expressed as $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$, where $g_1 \in Mat_k(\mathbb{R})$. Let g_+, g_- be matrices described in Lemma 3.12, corresponding to the blocks g_1 through g_4 and satisfying $g = g_+ g_-^{-1}$. Since $g.I = I$, it follows that $g_-.I = g_+.I$. Denote that $X_0 = X$ and $Z_0 = g.Z$, both being diagonal matrices. Then, $(g_+^{-1})^{\mathrm{T}}.Z = (g_-^{-1})^{\mathrm{T}}.Z_0$. Our goal is to show that the images of $Bis(X,Y)$ and $Bis(Y,Z)$ under the $(g_+^{-1})^{\mathrm{T}}$ action are disjoint, which can be expressed as:

$$(g_+.(X_0^{-1} - I))^{\perp} \cap (g_-.(Z_0^{-1} - I))^{\perp} = \varnothing. \tag{*}$$

For $X_0 = \mathrm{diag}(x_i)$ and $Z_0 = \mathrm{diag}(z_i)$, Lemma 3.11 implies that for any $i \leq k$ and $j > k$,

$$\frac{|x_j^{-1} - 1|}{|x_i^{-1} - 1|} \geq t \cdot \left(\frac{L-1}{n-1}\right)^t, \ \frac{|z_i^{-1} - 1|}{|z_j^{-1} - 1|} \geq t \cdot \left(\frac{L-1}{n-1}\right)^t.$$

Thus, there exist positive constants c_x and c_z such that for any $i \le k$ and $j > k$,

$$\begin{aligned} c_x(x_j^{-1} - 1) &\ge t \cdot \left(\frac{L-1}{n-1}\right)^t, \quad -1 \le c_x(x_i^{-1} - 1) < 0. \\ c_z(z_i^{-1} - 1) &\ge t \cdot \left(\frac{L-1}{n-1}\right)^t, \quad -1 \le c_z(z_j^{-1} - 1) < 0. \end{aligned} \tag{3.8}$$

Denote

$$h = (h_{ij}) = diag(I_k, O)g_+ + diag(O, I_{n-k})g_-,$$

then

$$h = \begin{pmatrix} (g_1^{-1})^{\mathrm{T}} & -(g_1^{-1})^{\mathrm{T}} g_3^{\mathrm{T}} \\ -g_4^{-1} g_3 & g_4^{-1} \end{pmatrix},$$

and h is decomposed as $h = h_a^{-1} h_b$, where

$$h_a = \begin{pmatrix} g_1^{\mathrm{T}} & O \\ O & g_4 \end{pmatrix}, \; h_b = \begin{pmatrix} I & -g_3^{\mathrm{T}} \\ -g_3 & I \end{pmatrix}.$$

As g_X is assumed to be the identity matrix, the angle between $\mathbf{e}_i$ and the i-th column vector of g_Z is at most θ, which implies that the diagonal elements of h_a are no less than $\cos\theta$. Moreover,

$$\sum_{j \ne i, j \le k} |g_{ij}| \le \sqrt{(k-1) \sum_{j \ne i, j \le k} g_{ij}^2} \le \sqrt{(k-1)} \sin\theta \le \sqrt{(n-2)} \sin\theta,$$

thus $\sigma_r(h_b), \sigma_c(h_b) \le 1 + \sqrt{n-2}\sin\theta$. By Lemma 3.13 and 3.14, we deduce that

$$\sigma_r(h), \sigma_c(h) \le \frac{1 + \sqrt{n-2}\sin\theta}{\cos\theta - \sqrt{n-2}\sin\theta}.$$

We establish the condition (*) by proving the positive definiteness of the linear combination $c_x \cdot g_+.(X_0^{-1} - I) + c_z \cdot g_-.(Z_0^{-1} - I)$. Let $c_x \cdot g_+.(X_0^{-1} - I) = (\xi_{ij})$ and $c_z \cdot g_-.(Z_0^{-1} - I) = (\zeta_{ij})$. For $i \le k$, we have the following inequalities:

$$\begin{aligned} &\xi_{ii} = \sum_{l \le k} h_{li}^2 (x_l^{-1} - 1) \ge -\sum_{l \le k} h_{li}^2, \\ &\sum_{j \ne i} |\xi_{ij}| \le \sum_{j \ne i,\ l \le k} |h_{li}||h_{lj}||x_l^{-1} - 1| \le \sum_{j \ne i,\ l \le k} |h_{li}||h_{lj}|, \\ &\zeta_{ii} = (z_i^{-1} - 1) + \sum_{l > k} h_{li}^2 (z_l^{-1} - 1) \ge t((L-1)/(n-1))^t - \sum_{l > k} h_{li}^2, \\ &\sum_{j \ne i} |\zeta_{ij}| \le \sum_{j \ne i,\ l > k} |h_{li}||h_{lj}||z_l^{-1} - 1| \le \sum_{j \ne i,\ l > k} |h_{li}||h_{lj}|. \end{aligned}$$

Hence,

$$\begin{aligned} \xi_{ii} + \zeta_{ii} &\ge t((L-1)/(n-1))^t - \sum_{l=1}^n h_{li}^2 \ge \left(\frac{1 + \sqrt{n-2}\sin\theta}{\cos\theta - \sqrt{n-2}\sin\theta}\right)^2 - \sum_{l=1}^n h_{li}^2 \\ &= \sigma_r(h)\sigma_c(h) - \sum_{l=1}^n h_{li}^2 \ge \sum_{l=1}^n \sigma_r(h)|h_{li}| - \sum_{l=1}^n h_{li}^2 \ge \sum_{l,j} |h_{lj}||h_{li}| - \sum_{l=1}^n h_{li}^2 \\ &= \sum_{j \ne i, 1 \le l \le n} |h_{li}||h_{lj}| \ge \sum_{j \ne i} |\xi_{ij} + \zeta_{ij}|. \end{aligned}$$

For $i > k$, the inequality $\xi_{ii} + \zeta_{ii} \ge \sum_{j \ne i} |\xi_{ij} + \zeta_{ij}|$ holds analogously. This implies that $c_x \cdot g_+.(X_0^{-1} - I) + c_z \cdot g_-.(Z_0^{-1} - I)$ is diagonally dominant and hence positive definite. Therefore, the condition (*) holds, implying that $Bis(X, Y)$ and $Bis(Y, Z)$ are disjoint. $\square$

3.5. *Proof of Theorem 2.6*

Let Γ be a discrete subgroup of $SL(3,\mathbb{R})$ and let $X \in \mathcal{P}(3)$. A facet of the Dirichlet-Selberg domain $D = DS(X,\Gamma)$ lies in the bisector $Bis(X,\gamma.X)$ for a certain $\gamma \in \Gamma$. We denote such a facet by F_γ. The existence of such facets is characterized by the following lemma:

LEMMA 3.15. *Let Γ be a discrete subgroup of $SL(n,\mathbb{R})$. Suppose that there exists a smooth function $g : \mathbb{R}^m \to SL(n,\mathbb{R})$ such that $\Gamma = g(\Lambda)$, where Λ is a discrete subset of $\mathbb{R}^m$, $\mathbf{0} \in \Lambda$, and $g(\mathbf{0}) = e$. For $A, X \in \mathcal{P}(n)$, define a function $s^g_{X,A} : \mathbb{R}^m \to \mathbb{R}$, $s^g_{X,A}(\mathbf{k}) = s(g(\mathbf{k}).X, A)$.*

Then for any $\mathbf{k}_0 \in \Lambda\backslash\{\mathbf{0}\}$, the facet $F_{g(\mathbf{k}_0)}$ of $DS(X,\Gamma)$ exists if and only if there exists a matrix $A \in \mathcal{P}(n)$ such that $\mathbf{0}$ and $\mathbf{k}_0$ are the only minimum points of $s^g_{X,A}|_\Lambda$.

Proof. The existence of the facet $F_{g(\mathbf{k}_0)}$ is equivalent to the existence of an interior point A of the facet. Moreover, $s^g_{X,A}$ for this interior point A satisfies the lemma requirements, and vice versa. $\square$

REMARK 1. Lemma 3.15 provides insights into the nature of Dirichlet-Selberg domains. Given $X \in \mathcal{P}(n)$ and $\Gamma < SL(n,\mathbb{R})$, the lemma implies the following:

- If for all but finitely many points $\mathbf{k} \in \Lambda$ and for every $A \in \mathcal{P}(n)$, the function $s^g_{X,A}|_\Lambda$ cannot be minimum at both $\mathbf{k}$ and $\mathbf{0}$, the Dirichlet-Selberg domain $DS(X,\Gamma)$ is finitely-sided.
- If there are infinitely many points $\mathbf{k} \in \Lambda$ such that $\mathbf{k}$ and $\mathbf{0}$ are the only two minimum points of $s^g_{X,A}|_\Lambda$ for a certain $A \in \mathcal{P}(n)$, the Dirichlet-Selberg domain $DS(X,\Gamma)$ is infinitely-sided.

We present a generalization of Lemma 3.15:

COROLLARY 3.16. *Let Γ, g, Λ and $s^g_{X,A}$ be as defined in Lemma 3.15. Suppose that there exists a matrix $A \in \mathcal{P}(n)$ and a finite subset $\Lambda_0 \subset \Lambda$ satisfying the following conditions:*

(i) *The point $\mathbf{0} \in \Lambda_0$.*

(ii) *There exists a nonzero point $\mathbf{k_0} \in \Lambda_0$ such that $s^g_{X,A}(\mathbf{k_0}) = s^g_{X,A}(\mathbf{0})$.*

(iii) *For any $\mathbf{k} \in \Lambda_0$, $s^g_{X,A}(\mathbf{k}) \le s^g_{X,A}(\mathbf{0})$; for any $\mathbf{k} \in \Lambda\backslash\Lambda_0$, $s^g_{X,A}(\mathbf{k}) > s^g_{X,A}(\mathbf{0})$.*

Then the Dirichlet-Selberg domain $DS(X,\Gamma)$ has a facet $F_{g(\mathbf{k})}$ for at least one element $\mathbf{k} \in \Lambda_0\backslash\{\mathbf{0}\}$.

Proof. Let $\Lambda_0 = \{\mathbf{0}, \mathbf{k}_0, \mathbf{k}_1, \dots, \mathbf{k}_r\}$, where the elements are ordered as

$$s^g_{X,A}(\mathbf{k}_0) \ge s^g_{X,A}(\mathbf{k}_1) \ge \cdots \ge s^g_{X,A}(\mathbf{k}_r).$$

Define

$$\Lambda'_i := (\Lambda\backslash\Lambda_0) \cup \{\mathbf{0}, \mathbf{k}_0, \dots, \mathbf{k}_i\},\ i = 0, \dots, r.$$

The following statement is evident by induction on i, utilizing Lemma 3.15:

(*) The Dirichlet-Selberg domain $DS(X, g(\Lambda'_i))$ contains a facet $F_{g(\mathbf{k}_j)}$ for a certain $j \in \{0, \dots, i\}$.

When $i = r$, (*) concludes this corollary. $\square$

The proof of Theorem 2.6 comprises a series of assertions, divided into finitely-sided and infinitely-sided parts. For clarity, we consistently denote the (i,j) entry of X^{-1} and A by x^{ij} and a_{ij}, respectively. We denote the generator of cyclic groups by γ, and the two generators of 2-generated groups by γ_1 and γ_2, as listed in Proposition 2.5.

Proof of Theorem 2.6 for finitely-sided cases.

Cyclic group of type (1). We interpret the group $\Gamma = \langle\gamma\rangle$ as the image of $\mathbb{Z}$ under the function $g(k) = \gamma^k$, $\forall k \in \mathbb{R}$. The function $s^g_{X,A}$ described in Lemma 3.15 becomes a quadratic polynomial with a positive leading coefficient:

$$s^g_{X,A}(k) = x^{11}a_{22}k^2 + 2(x^{11}a_{12} + x^{12}a_{22} + x^{13}a_{23})k + const.$$

Thus, if $s^g_{X,A}|_{\mathbb{Z}}$ attains its minimum at $k = 0$, the other possible minimum point is either $k = -1$ or $k = 1$. The remark following Lemma 3.15 implies that $D_S(X, \Gamma)$ is two-sided for any $X \in \mathcal{P}(3)$.

Cyclic group of type (3). Similarly, we interpret the group $\Gamma = \langle\gamma\rangle$ as a one-parameter family, $g(k) = \gamma^k = diag(e^{rk}, e^{sk}, e^{tk})$, where $r + s + t = 0$, and $r, s, t \neq 0$. Without loss of generality, we assume that $r \geq s > 0 > t$. The function $s^g_{X,A}$ described in Lemma 3.15 becomes:

$$s^g_{X,A}(k) = x^{11}a_{11}e^{-2rk} + x^{22}a_{22}e^{-2sk} + x^{33}a_{33}e^{-2tk} + 2x^{23}a_{23}e^{rk} + 2x^{13}a_{13}e^{sk} + 2x^{12}a_{12}e^{tk}.$$

Since x^{ii} and $a_{ii} > 0$ for $i = 1, 2, 3$, there exists a unique $k_c \in \mathbb{R}$ such that

$$\sqrt{x^{11}a_{11}}e^{-rk_c} + \sqrt{x^{22}a_{22}}e^{-sk_c} = \sqrt{x^{33}a_{33}}e^{-tk_c}.$$

Hence, $s^g_{X,A}(k) = c \cdot f(k - k_c)$, where

$$f(k) = e^{2(r+s)k} + 2p\alpha_{13}e^{sk} + 2(1-p)\alpha_{23}e^{rk} + p^2e^{-2rk} + (1-p)^2e^{-2sk} + 2p(1-p)\alpha_{12}e^{-(r+s)k},$$

and

$$c = x^{33}a_{33}e^{-2tk_c} > 0,\ p = \frac{\sqrt{x^{11}a_{11}}e^{-rk_c}}{\sqrt{x^{33}a_{33}}e^{-tk_c}} \in (0,1),\ \alpha_{ij} = \frac{x^{ij}a_{ij}}{\sqrt{x^{ii}x^{jj}a_{ii}a_{jj}}},$$

with $|\alpha_{ij}| < \xi := \max_{i \neq j} \frac{|x^{ij}|}{\sqrt{x^{ii}x^{jj}}} < 1$.

For any $(p, \alpha_{12}, \alpha_{13}, \alpha_{23}) \in [0,1] \times [-\xi, \xi]^3$, there exists $N > 0$ such that

$$f'(n; p, \alpha_{ij}) > 0,\ \forall n > N;\ f'(n; p, \alpha_{ij}) < 0,\ \forall n < -N.$$

This is shown by considering the cases $p = 0$, $p = 1$, and $0 < p < 1$ separately; for either case, the leading terms as $k \to \infty$ and $k \to -\infty$ have positive coefficients.

Thus, the minimum points of f lie between $-N$ and N. The compactness of the region implies that N exists uniformly for all tuples $(p, \alpha_{12}, \alpha_{13}, \alpha_{23}) \in [0,1] \times [-\xi, \xi]^3$. Consequently, if $k = 0$ and $k = k_0$ are the only minimum points of $s^g_{X,A}|_{\mathbb{Z}}$, then $|k_0| < 2(N+1)$. Lemma 3.15 implies that $D_S(X, \Gamma)$ is finitely-sided for any $X \in \mathcal{P}(3)$.

Cyclic group of type (5). Similarly, we interpret the group $\Gamma = \langle\gamma\rangle$ as a one-parameter family, $g(k) = \gamma^k$, and the function $s^g_{X,A}$ becomes

$$\begin{aligned} s^g_{X,A}(k) &= x^{33}a_{33}e^{4sk} + (2x^{13}a_{13} + 2x^{23}a_{23} - 2ke^{-s}x^{23}a_{13})e^{sk} \\ &+ (x^{11}a_{11} + x^{22}a_{22} + 2x^{12}a_{12} - 2ke^{-s}(x^{12}a_{11} + x^{22}a_{12}) + k^2e^{-2s}x^{22}a_{11})e^{-2sk}, \end{aligned}$$

where $s \neq 0$; assume that $s > 0$ without loss of generality. Similarly to the preceding case, we can interpret $s^g_{X,A}(k) = c \cdot f(k - k_c)$ for a certain $k_c \in \mathbb{R}$, where

$$\begin{aligned} f(n) &= e^{4sn} + (2\alpha_{13}p + 2\alpha_{23}q - 2\beta_3(1-p-q)n)e^{sn} \\ &+ (p^2 + q^2 + 2\alpha_{12}pq - 2(\beta_1 p + \beta_2 q)(1-p-q)n + (1-p-q)^2n^2)e^{-2sn}, \end{aligned}$$

with $p, q > 0$, $p + q < 1$, $|\alpha_{ij}|, |\beta_1|, |\beta_3| < \xi$ and $|\beta_2| < 1$, all dependent on x_{ij}, a_{ij}, s, and k_c.

Similarly to the preceding case, the compactness of the region $\{(p,q) | p, q \geq 0, p + q \leq 1\} \times [-\xi, \xi]^5 \times [-1, 1]$ implies the existence of a number $N > 0$, such that $|k_0| < 2(N+1)$ if $k = 0$ and $k = k_0$ are the only minimum points of $s^g_{X,A}|_{\mathbb{Z}}$. Lemma 3.15 implies that $D_S(X, \Gamma)$ is finitely-sided for any $X \in \mathcal{P}(3)$.

Two-generated group of type (1). We interpret the group Γ as a two-parameter family, $g(k,l) = \gamma_1^k \gamma_2^l$, $k, l \in \mathbb{Z}$. Computation suggests that:

$$s^g_{X,A}(k,l) = a_{11}(x^{22}(k-k_c)^2 + 2x^{23}(k-k_c)(l-l_c) + x^{33}(l-l_c)^2) + const,$$

where k_c, l_c depend on a_{ij} and x^{ij}. Since $x^{22}x^{33} > (x^{23})^2$, the level curves of $s^g_{X,A}$ are ellipses centered at (k_c, l_c), with same eccentricities dependent on X. If such a level curve surrounds two points in $\mathbb{Z}^2$ and excludes all other integer points, its major axis length is bounded by a constant dependent on X. Therefore, there are only finitely many choices of $(k_0, l_0) \in \mathbb{Z}^2$, such that $(0,0)$ and (k_0, l_0) are the only minimum points of $s^g_{X,A}|_{\mathbb{Z}^2}$. It follows that $DS(X,\Gamma)$ is finitely-sided for any $X \in \mathcal{P}(3)$.

Two-generated group of type (4). We interpret the group Γ as a two-parameter family, $g(k,l,m) = diag(e^k, e^l, e^m)$, where the domain of g is the plane:

$$\{(k,l,m) \in \mathbb{R}^3 | k + l + m = 0\},$$

and the preimage of Γ is $\Lambda = \mathbb{Z}(r,s,t) \oplus \mathbb{Z}(r',s',t')$. The function $s^g_{X,A}$ is given by

$$\begin{aligned}&(x^{11}a_{11})e^{2k} + (x^{22}a_{22})e^{2l} + (x^{33}a_{33})e^{2m} + (2x^{23}a_{23})e^{-k} + (2x^{13}a_{13})e^{-l} + (2x^{12}a_{12})e^{-m} \\ &= c(e^{2(k-k_c)} + e^{2(l-l_c)} + e^{2(m-m_c)} + 2\alpha_{23}e^{-(k-k_c)} + 2\alpha_{13}e^{-(l-l_c)} + 2\alpha_{12}e^{-(m-m_c)}),\end{aligned}$$

for some constants c, k_c, l_c, m_c, α_{12}, α_{13}, and α_{23} dependent on x^{ij} and a_{ij}. Moreover, $c > 0$, $|\alpha_{ij}| < \xi := \max_{i \neq j} \frac{|x^{ij}|}{\sqrt{x^{ii}x^{jj}}}$, and $k_c + l_c + m_c = 0$. Let $d = d(k,l,m)$ represent the Euclidean distance between (k_c, l_c, m_c) and (k,l,m) divided by $\sqrt{6}/2$, then:

$$2(1-\xi)e^d - 4\xi e^{-d/2} + e^{-2d} := f_-(d) \leq s^g_{X,A}(k,l,m)/c \leq f_+(d) := e^{2d} + 4\xi e^{d/2} + 2(1+\xi)e^{-d},$$

where the lower bound is attained when $\alpha_{ij} = -\xi$ and $(k-k_c, l-l_c, m-m_c) = (-d, d/2, d/2)$, while the upper bound is attained when $\alpha_{ij} = \xi$ and $(k-k_c, l-l_c, m-m_c) = (d, -d/2, -d/2)$. Moreover, $\lim_{d\to\infty} f_-(d) = \infty$. For each level curve of $s^g_{X,A}$, the inequality implies that its diameter D is controlled by its inscribed radius ρ via $f_-(D/2) \leq f_+(\rho)$. Similarly to the preceding case, there are only finitely many choices of $(k_0, l_0, m_0) \in \Lambda$, such that $(0,0,0)$ and (k_0, l_0, m_0) are the only minimum points of $s^g_{X,A}|_\Lambda$. It follows that $DS(X,\Gamma)$ is finitely-sided for any $X \in \mathcal{P}(3)$. □

We now consider the cases when the Dirichlet-Selberg domain $DS(X,\Gamma)$ is infinitely-sided for a generic choice of $X \in \mathcal{P}(3)$. In the following proofs, we drop the requirement $\det(A) = 1$, as this condition can be regained by rescaling the matrix A whenever A is positive definite.

Proof of Theorem 2.6 for infinitely-sided cases.

Cyclic group of type (2). We interpret the cyclic group Γ as a one-parameter family, $g(k) = \gamma^k$, where $k \in \mathbb{Z}$. The function $s^g_{X,A}$, described in Lemma 3.15, is expressed as a quartic polynomial in k:

$$\begin{aligned}&s^g_{X,A}(k) = (x^{33}a_{11}/4)k^4 + \left(-x^{33}a_{12} + \left(x^{33}/2 - x^{23}\right)a_{11}\right)k^3 \\ &+ \left(x^{33}a_{13} + x^{33}a_{22} + (3x^{23} - x^{33})a_{12} + \left(x^{33}/4 - x^{23} + x^{13} + x^{22}\right)a_{11}\right)k^2 \\ &+ \left(-2x^{33}a_{23} + (x^{33} - 2x^{23})a_{13} - 2x^{23}a_{22} + (x^{23} - 2x^{13} - 2x^{22})a_{12} + (x^{13} - 2x^{12})a_{11}\right)k \\ &+ (x^{33}a_{33} + x^{22}a_{22} + x^{11}a_{11} + 2x^{23}a_{23} + 2x^{13}a_{13} + 2x^{12}a_{12}).\end{aligned}$$

For any $X \in \mathcal{P}(3)$ and any $k_0 \in \mathbb{Z}$, our goal is to find a positive definite matrix A such that

$$s^g_{X,A}(k) = k^2(k-k_0)^2 + const,$$

ensuring that $k = 0$ and $k = k_0$ are the only (global) minimum points of $s^g_{X,A}$.

The entries a_{11} and a_{12} are determined by comparing the k^4 and k^3 coefficients, following by choosing a_{22} sufficiently large such that $a_{11}a_{22} - a_{12}^2 > 0$. Subsequently, the entries a_{13} and

a_{23} are determined by solving a linear equation system derived from the coefficients of k^2 and k^1, yielding a unique solution. Finally, let a_{33} be sufficiently large so that A is positive definite.

These steps result in a matrix $A \in F_{\gamma^k}$. Lemma 3.15 implies that $D_S(X, \Gamma)$ is infinitely-sided for any $X \in \mathcal{P}(3)$.

Cyclic group of type (4). We interpret the cyclic group Γ as a one-parameter family, $g(k) = \gamma^k$, $k \in \mathbb{Z}$. The function $s^g_{X,A}$ is expressed as:

$$s^g_{X,A}(k) = a_{22}x^{22}e^{2sk} + 2a_{23}x^{23}e^{sk} + 2a_{13}x^{13}e^{-sk} + a_{11}x^{11}e^{-2sk} + const.$$

For any $X \in \mathcal{P}(3)$ with $x^{13}x^{23} \neq 0$ and any $k_0 \in \mathbb{Z}$, we can find a positive definite matrix A such that

$$s^g_{X,A}(k) = e^{2sk} - 2(e^{sk_0} + 1)e^{sk} - 2e^{sk_0}(e^{sk_0} + 1)e^{-sk} + e^{2sk_0}e^{-2sk} + const,$$

similarly to the preceding case. This function has two global minimum points, namely $k = 0$ and $k = k_0$. Lemma 3.15 implies that $D_S(X, \Gamma)$ is infinitely-sided whenever X does not belong to the proper Zariski closed subset $\{X = (x^{ij})^{-1} \in \mathcal{P}(3) | x^{13}x^{23} = 0\}$.

Two-generated group of type (2). We interpret the group Γ as a two-variable family, $g(k, l) = \gamma_1^k\gamma_2^l$, $(k, l) \in \mathbb{Z}^2$. The function $s^g_{X,A}$ is expressed as

$$s^g_{X,A}(k, l) = x^{33}(a_{22}(k - k_c)^2 + 2a_{12}(k - k_c)(l - l_c) + a_{11}(l - l_c)^2) + const,$$

and its level curves are ellipses with the center (k_c, l_c) dependent on x^{ij} and a_{ij}. Unlike the two-generated groups of type (i), the eccentricities of these ellipses depend on A. Specifically, for any coprime pair $(k_0, l_0) \in \mathbb{Z}^2$ and arbitrarily small $\epsilon > 0$, we can choose the matrix A so that the equation holds:

$$s^g_{X,A}(k, l) = \epsilon^2(k_0(k - k_0/2) + l_0(l - l_0/2))^2 + (l_0(k - k_0/2) - k_0(l - l_0/2))^2 + const. \quad (3.9)$$

Entries a_{11}, a_{12} and a_{22} are uniquely determined by comparing the k^2, kl and l^2 coefficients and guarantee that $a_{11}a_{22} > a_{12}^2$. Furthermore, a_{13} and a_{23} are uniquely determined by letting $(k_c, l_c) = (k_0/2, l_0/2)$. Finally, let a_{33} be sufficiently large so that A is positive definite.

A particular level curve of such $s^g_{X,A}$ has its major axis as the line segment between $(0, 0)$ and (k_0, l_0), and its minor axis length be ϵ times the length of the major axis. Since k_0 and l_0 are coprime, the ellipse excludes all other points in $\mathbb{Z}^2$ when ϵ is sufficiently small. By Lemma 3.15, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is infinitely-sided for any $X \in \mathcal{P}(3)$.

Two-generated group of type (3). We interpret the group Γ as a two-variable family:

$$g(k, l) = \begin{pmatrix} 1 & -k & k^2 - l \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix}, \ \forall(k, l) \in \Lambda = \Lambda(a, b),$$

where

$$\Lambda(a, b) = \left\{(k, l) \,\middle|\, k = x + ay, l = \frac{1}{2}\left(a^2(y^2 - y) + 2axy + 2by + x^2 - x\right),\ (x, y) \in \mathbb{Z}\right\}$$

is a discrete subset of $\mathbb{R}^2$. The function $s^g_{X,A}$ is expressed as

$$\begin{aligned} s^g_{X,A}(k, l) &= (a_{11}x^{22} + 2a_{12}x^{23} + a_{22}x^{33})(k - k_c)^2 \\ &+ 2(a_{11}x^{23} + a_{12}x^{33})(k - k_c)(l - l_c) + (a_{11}x^{33})(l - l_c)^2 + const, \end{aligned}$$

where k_c and l_c depend on a_{ij} and x^{ij}.

We claim that for sufficiently small $\delta > 0$, there exists $\epsilon = \epsilon(\delta; X) > 0$, such that $\epsilon = O(\delta^2)$ as $\delta \to 0$, and for any $(k_0, l_0) \in \Lambda$ with $|k_0/l_0| = \delta$, there exists a positive definite matrix A satisfying Equation 3.9.

Comparison of the k^2, kl and l^2 coefficients yields a linear equation system in unknowns of a_{11}, a_{12} and a_{22}, which admits a unique solution. Given this solution, the positive definite condition $a_{11}a_{22} > a_{12}^2$ holds if

$$\epsilon = \epsilon(\delta) := \frac{\sqrt{x^{22}x^{33} - {x^{23}}^2}}{x^{33}}\delta^2 + O(\delta^4).$$

Setting $(k_c, l_c) = (k_0/2, l_0/2)$ yields a linear equation system in unknowns a_{13} and a_{23} with an invertible coefficient matrix, uniquely determining a_{13} and a_{23}. Finally, let a_{33} be sufficiently large so that A is positive definite. For such a matrix A, a particular level curve of $s^g_{X,A}$ is an ellipse whose major axis is between $(0,0)$ and (k_0, l_0), and whose minor axis length is $\epsilon = O((k_0/l_0))^2)$ times the length of the major axis.

We address two cases based on whether the entry a of the generator γ_2 is rational. If $a \in \mathbb{Q}$, we assume that $a = p/q$, where (p, q) are coprime. The first components of points in Λ take values in $(1/q)\mathbb{Z}$, and

$$\Lambda \cap \{(k_0, l_0) | k_0 = 1/q\} = \{(1/q, l_{0(n)}) | l_{0(n)} = (a(a-1) - 2b)qn + l_{0(0)},\ n \in \mathbb{Z}\},$$

where $l_{0(0)}$ is a constant depending on a and b. By applying our construction of matrix A to $(1/q, l_{0(n)})$, we derive level curves surrounding $(0,0)$ and $(1/q, l_{0(n)})$. Let $\delta_n = (1/q)/l_{0(n)}$ and $\epsilon_n = \epsilon(\delta_n)$; then $\delta_n = O(n^{-1})$, and thus $\epsilon_n = O(n^{-2})$ as $n \to \infty$. Elementary computation implies that the level curve we constructed for $(k_0, l_0) = (1/q, l_{0(n)})$ lies between the lines

$$k = \frac{1 \pm \sqrt{1 + q^2(l_{0(n)})^2\epsilon_n^2}}{2q} = \frac{1 \pm (1 + O(n^{-2}))}{2q}.$$

Thus, it is disjoint from the lines $k = 2/q$ and $k = -1/q$ for large n. Moreover, its other intersection with the line $k = 0$ is

$$\left(0, \frac{l_{0(n)}(l_{0(n)}^2 + q^{-2})\epsilon_n^2}{q^{-2} + l_{0(n)}^2\epsilon_n^2}\right) = (0, O(n^{-1})),$$

which can be arbitrarily close to $(0,0)$ for large n. Consequently, the level curve excludes all other points in Λ for sufficiently large n. By Lemma 3.15, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is infinitely-sided for any $X \in \mathcal{P}(3)$.

If $a \notin \mathbb{Q}$, there are points (k, l) in Λ such that k is arbitrarily close to 0 while l is arbitrarily large. Therefore, we can choose points (k_i, l_i), $i = 1, 2, \ldots$ inductively, such that the level curve of $s^g_{X,A}$ we constructed previously for (k_i, l_i) excludes all points in $\Lambda \backslash \{(0,0)\}$ that are surrounded by either of the level curves for (k_j, l_j), $j < i$. By Corollary 3.16, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is infinitely-sided for any $X \in \mathcal{P}(3)$.

Two-generated group of type (5). We interpret the group Γ as a two-variable family:

$$g(k, l) = \begin{pmatrix} e^{-k} & -le^{-k} & 0 \\ 0 & e^{-k} & 0 \\ 0 & 0 & e^{2k} \end{pmatrix},\ \forall (k, l) \in \Lambda = \mathbb{Z}(t, 1) \oplus \mathbb{Z}(s, a) \subset \mathbb{R}^2,$$

where $(s, t) \neq (0, 0)$ and $a \in \mathbb{R}$. The function $s^g_{X,A}$ is expressed as follows:

$$\begin{aligned} s^g_{X,A}(k, l) &= e^{2k}(a_{11}x^{11} + 2a_{12}x^{12} + a_{22}x^{22} + 2l(a_{11}x^{12} + a_{12}x^{22}) + l^2 a_{11}x^{22}) \\ &+ 2e^{-k}(a_{13}x^{13} + a_{23}x^{23} + la_{13}x^{23}) + e^{-4k}a_{33}x^{33}. \end{aligned}$$

We claim that if $x^{23} \neq 0$, then for any $(k_0, l_0) \in \Lambda$ where $k_0 \neq 0$, there exists a point $A \in \mathcal{P}(3)$, such that:

- A level curve of $s^g_{X,A}$ is connected and passes through $(0,0)$ and (k_0, l_0).
- The level curve lies between the lines $k = 0$ and $k = k_0$, and is tangent to these lines at $(0,0)$ and (k_0, l_0), respectively.

Indeed, the level curve $s^g_{X,A}(k, l) = c$ is the union of graphs of the following functions:

$$l = L_\pm(e^{-k}; c) = L_0(e^{-3k}; c) \pm \sqrt{L_1(e^{-k}; c)},$$

where L_0 is linear in e^{-3k}, and L_1 is a degree 6 polynomial in e^{-k}.

We set $a_{11} = 1$. The entries a_{12} and a_{13} are uniquely determined by setting $L_0(1) = 0$ and $L_0(e^{-3k_0}) = l_0$. The entries a_{23} and a_{33} are uniquely determined by setting $L_1(1) = L_1(e^{-k_0}) = 0$ and depend on k_0, l_0, x^{ij}, c and a_{22}. Under these solutions, $\det(A)$ forms a quadratic polynomial in c, with the c^2 coefficient:

$$-\frac{\left(1+e^{k_0}\right)^2\left(1+e^{2k_0}\right)^2}{4\left(e^{k_0}+e^{2k_0}+1\right)^2 {x^{23}}^2} < 0.$$

Setting c to be the maximum point of $\det(A)$, this determinant becomes a quadratic polynomial in a_{22}, with the a_{22}^2 coefficient

$$\frac{e^{4k_0}\left(e^{2k_0}{x^{23}}^2+\left(1+e^{k_0}\right)^2\left(1+e^{2k_0}\right)x^{22}x^{33}\right)}{\left(1+e^{k_0}\right)^2\left(1+e^{2k_0}\right)^2 {x^{33}}^2} > 0.$$

Therefore, A is positive definite when a_{22} is sufficiently large. Moreover, one verifies that $t = 1$ and $t = e^{-k_0}$ are the only positive zeroes of $L_1(t)$, thus the level curve is connected.

We similarly discuss the two cases based on whether t/s is rational. If $t/s = p/q \in \mathbb{Q}$, where (p, q) are coprime, the first components of points in Λ take values in $(s/q)\mathbb{Z}$, and there are infinitely many points in $\Lambda \cap \{k = s/q\}$. The level curve we constructed for such a point $(s/q, l_0)$ lies between the lines $k = 0$ and $k = s/q$, thus it excludes all points in Λ other than $(0, 0)$ and $(s/q, l_0)$. By Lemma 3.15, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is infinitely-sided for any X not belonging to the proper Zariski closed subset $\{X = (x^{ij})^{-1} \in \mathcal{P}(3) | x^{23} = 0\}$.

If $t/s \notin \mathbb{Q}$, we can utilize Corollary 3.16 and prove that $DS(X, \Gamma)$ is infinitely-sided for any X not belonging to the aforementioned proper Zariski closed subset, similarly to the proof for two-generated groups of type (3). □

3.6. *Proof of Theorem 2.7*

We first proof Case (i) of the Theorem, where n is assumed to be even.

Proof of Theorem 2.7, Case (i). Denote the eigenvalues of A_i by

$$\lambda_{i,1} \geq \cdots \geq \lambda_{i,n/2} > 1 > \lambda_{i,n/2+1} \geq \cdots \geq \lambda_{i,n} > 0,\ i = 1, \ldots, k,$$

and let $\mathbf{v}_{i,j}$ be the corresponding eigenvector for $j = 1, \ldots, n$. Recall that $C^+_{A_i} = span(\mathbf{v}_{i,1}, \ldots, \mathbf{v}_{i,n/2})$, and $C^-_{A_i} = span(\mathbf{v}_{i,n/2+1}, \ldots, \mathbf{v}_{i,n})$. We claim that there exists an integer M satisfying the following conditions:

- For any real numbers $m_i^\pm \geq M$, $i = 1, \ldots, k$, the $2k$ bisectors $Bis(I, A_i^{m_i^+}.I)$, $Bis(I, A_i^{-m_i^-}.I)$ are pairwise disjoint.
- For each bisector σ among the $2k$ ones, the center I of the Dirichlet-Selberg domain and the other $(2k-1)$ bisectors lie in the same connected component of $\sigma^c = \mathcal{P}(n)\backslash\sigma$.

These claims will ensure that $\Gamma = \langle A_1^M, \ldots, A_k^M \rangle$ is Schottky. The first claim follows because

$$\lim_{m\to\infty} ||(A_i^{\mp m})^{\mathrm{T}}\mathbf{x}|| = \begin{cases} 0, & \forall \mathbf{x} \in C^\pm_{A_i}, \\ \infty, & \text{otherwise}, \end{cases}$$

which implies the existence of a positive number M such that for any $m_i^\pm \geq M$, certain positive linear combinations of any two among the $2k$ functions

$$||(A_i^{\mp m_i^\pm})^{\mathrm{T}}\mathbf{x}||^2/||\mathbf{x}||^2 - 1,\ i = 1, \ldots, k,$$

defined on the compact space $\mathbb{R}\mathbf{P}^{n-1}$, are positive. Utilizing Lemma 3.6, we deduce that the $2k$ bisectors

$$Bis(I, A_i^{\pm m_i^\pm}.I) = ((A_i^{\mp m_i^\pm})^{\mathrm{T}}.I - I)^\perp$$

are pairwise disjoint.

To prove our second claim, assume the opposite: there exist bisectors σ_1 and σ_2 among the $2k$ bisectors, such that σ_2 and the center I lie in different components of σ_1^{c}. Without loss of generality, suppose that $\sigma_1 = Bis(A_1^{m_1}.I, I)$ and $\sigma_2 = Bis(A_2^{m_2}.I, I)$. Fix a point $X \in \sigma_2$; as $m \to \infty$, X and I will be in the same component of $Bis(A_1^m.I, I)^{\mathsf{c}}$. Thus, a real number $m_1' > m_1 \geq M$ exists such that $X \in Bis(A_1^{m_1'}.I, I)$, contradicting our first claim. $\square$

We proceed to Case (ii), where n is assumed to be odd.

Proof of Theorem 2.7, Case (ii). Assume the opposite that $\Gamma = \langle A_1, \dots, A_k \rangle < SL(n, \mathbb{R})$ is Schottky, and none of the eigenvalues of these generators has an absolute value of 1. Without loss of generality, we can assume that the center of the Dirichlet-Selberg domain is $X = I$, after conjugating these generators.

We extend the notions of attracting and repulsing subspaces:

$$C^+_{A_i,\mathbb{C}} = span_{\mathbb{C},|\lambda_{i,j}|>1}(\mathbf{v}_{i,j}),\ C^-_{A_i,\mathbb{C}} = span_{\mathbb{C},|\lambda_{i,j}|<1}(\mathbf{v}_{i,j}),$$

where $\mathbf{v}_{i,j} \in \mathbb{C}^n$ is the eigenvector of A_i^{T} associated with the eigenvalue $\lambda_{i,j}$. As n is odd, either $\dim_{\mathbb{C}}(C^+_{A_i,\mathbb{C}}) \geq (n+1)/2$ or $\dim_{\mathbb{C}}(C^-_{A_i,\mathbb{C}}) \geq (n+1)/2$; assume the former for all i without loss of generality. We deduce that

$$C^+_{A_1,\mathbb{C}} \cap C^+_{A_2,\mathbb{C}} \backslash \{\mathbf{0}\} \neq \varnothing.$$

On the one hand, for any $m \in \mathbb{N}$, the bisectors $Bis(A_1^m.I, I)$ and $Bis(A_2^m.I, I)$ are disjoint, following that Γ is a Schottky group with a ridge-free Dirichlet-Selberg domain centered at I.

On the other hand, we aim to derive a contradiction by showing that the bisectors $Bis(A_1^m.I, I)$ and $Bis(A_2^m.I, I)$ intersect for sufficiently large $m \in \mathbb{N}$. Take nonzero vectors

$$\mathbf{v} \in C^+_{A_1,\mathbb{C}} \cap C^+_{A_2,\mathbb{C}},\ \mathbf{w} \in (C^+_{A_1,\mathbb{C}} \cup C^+_{A_2,\mathbb{C}})^{\mathsf{c}}.$$

Similarly to the proof of Case (i), we establish that

$$\mathbf{w}^*((A_1^m.I)^{-1} - I)\mathbf{w},\ \mathbf{w}^*((A_2^m.I)^{-1} - I)\mathbf{w} > 0,$$

for sufficiently large m. Furthermore,

$$\mathbf{v}^*(A_1^m.I)^{-1}\mathbf{v} = ||(A_1^{-m})^{\mathrm{T}}\mathbf{v}||^2 = ||\varphi^m(\mathbf{v})||^2 \leq ||\varphi^m||^2 \cdot ||\mathbf{v}||^2,$$

where φ represents the restriction of the linear transformation $(A_1^{-1})^{\mathrm{T}}$ to the A_1^{T}-invariant subspace $C^+_{A_1,\mathbb{C}}$ of $\mathbb{C}^n$, whose spectral radius is less than 1. Gelfand's theorem implies that $\lim_{m\to\infty} ||\varphi^m|| = 0$; a similar assertion holds for A_2. Thus,

$$\mathbf{v}^*((A_1^m.I)^{-1} - I)\mathbf{v},\ \mathbf{v}^*((A_2^m.I)^{-1} - I)\mathbf{v} < 0,$$

for sufficiently large m.

These inequalities imply that the pencil

$$((A_1^m.I)^{-1} - I, (A_2^m.I)^{-1} - I)$$

is indefinite for sufficiently large m. Following Lemma 3.6, the bisectors $Bis(A_1^m.I, I)$ and $Bis(A_2^m.I, I)$ intersect for sufficiently large m, a contradiction. $\square$

Acknowledgements. I would like to express my sincere gratitude to Michael Kapovich. His insightful discussion with me shaped the main problems addressed, and his thorough inspection of my draft enriched the quality of this article. I am grateful for his dedication and support throughout the writing process.

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