SCHOTTKY PAIRS ON TREES VIA CONTINUED FRACTIONS AND AXIAL GEOMETRY

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Abstract. We give a complete criterion for when two hyperbolic automorphisms of a tree generate a free, discrete subgroup. The decision depends only on three geometric invariants: the translation lengths of the generators and the length of overlap of their axes. This data is organized using the continued-fraction expansion of the translation-length ratio. We extend the result to weighted trees, allowing arbitrary positive real translation lengths under local finiteness. In the irrational case, the exceptional configurations are shown to correspond precisely to the gap lengths in the three-gap theorem.

1. Introduction

This paper is concerned with the geometry and algebra of free, discrete subgroups generated by pairs of hyperbolic automorphisms of trees. Our approach is geometric and reduction-based: we encode a generating pair through their translation lengths and the configuration of their axes, and seek criteria that determine whether the resulting subgroup is free of rank two and, in that case, whether it is generated by a Schottky pair.

The classical motivation arises from the theory of Fuchsian and Kleinian groups, i.e. discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$, which act by isometries on the hyperbolic planes $\mathbb{H}^2$ and $\mathbb{H}^3$. Two-generator free subgroups (Schottky groups) in this setting can often be detected by geometric inequalities involving traces of the generating matrices and angles or distances between their axes; see, e.g. Jørgensen's inequality [7], which gives a necessary condition for discreteness, or the sharp length–angle criteria of Rosenberger [12]. While these results are well-understood in the archimedean setting, the corresponding problem over non-archimedean fields remains far less explicit.

This motivates the study of analogous phenomena in the realm of trees. When K is a non-archimedean local field, the group $\mathrm{PSL}(2,K)$ acts by isometries on its associated Bruhat–Tits tree, and questions of discreteness, freeness, and reduction become intertwined with the geometry of that action. More generally, one can view $\mathrm{Aut}(X)$, the automorphism group of a locally finite tree X, as a combinatorial analogue of $\mathrm{Isom}(\mathbb{H}^n)$ for the purpose of studying non-archimedean Schottky groups.

The overarching problem of understanding free discrete tree actions has deep roots in the study of p-adic groups, combinatorial geometry, and number theory. Such actions play a central role in the structure of rank-one p-adic Lie groups, where discrete free subgroups serve as convex-cocompact lattices—often arising as fundamental groups of Mumford curves

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or p-adic Drinfeld modular varieties. A foundational result of Lubotzky [9] asserts that any finitely generated, torsion-free, discrete subgroup of $\mathrm{Aut}(X)$ is automatically free and Schottky [9, Proposition 1.7], highlighting a rigidity phenomenon: while discreteness forces freeness in this context, detecting when a given generating pair yields a Schottky subgroup remains a subtle geometric problem. These issues are closely tied to expansion, spectral theory, and invariant measures, as developed in Lubotzky's monograph [10]. Beyond algebraic and geometric rigidity, Schottky tree actions also serve as uniformizing groups for p-adic analytic curves such as Mumford curves (see [3]), and their structural and spectral properties govern the fractal geometry of their limit sets. In joint work with Hubbard [6], the second-named author showed that the Hausdorff dimension of such limit sets can be computed explicitly using transfer operators. These connections underscore the need for a precise geometric criterion describing when a pair of hyperbolic automorphisms generates a Schottky subgroup of $\mathrm{Aut}(X)$. Yet, a precise geometric criterion characterizing when a given pair of hyperbolic automorphisms generates a free (or Schottky) subgroup is still lacking. With this in mind, we are interested in the following question.

Question. Let $\gamma_1, \gamma_2 \in \mathrm{Aut}(X)$ be hyperbolic automorphisms of a locally finite tree X, with translation lengths m_1 and m_2, and suppose their axes A_{γ_1} and A_{γ_2} intersect. Assume that the subgroup $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ is discrete. Under what conditions is Γ free of rank two? If Γ is free, can one find a Schottky pair $\gamma_{\mathrm{a}}, \gamma_{\mathrm{b}}$ generating Γ?

This question plays a central role in both classical and p-adic geometric group theory. In the setting of $\mathrm{SL}_2(K)$ acting on the Bruhat–Tits tree, Conder [1] developed an algorithmic reduction procedure using Nielsen moves to determine either that a word is elliptic or that the pair satisfies a Schottky ping–pong hypothesis. His approach yields a constructive membership criterion for two-generator free subgroups of $\mathrm{SL}_2(K)$.

Our aim here is different but complementary. Rather than solving the membership problem, we work in the general geometric setting of $\mathrm{Aut}(X)$ and give a complete structural classification of when a pair of hyperbolic automorphisms generates a free discrete subgroup, expressed entirely in terms of translation lengths and the length

$$l = \ell(A_{\gamma_1} \cap A_{\gamma_2}), \tag{1.1}$$

of intersection of their axes (with $l = 0$ in the disjoint case). Up to conjugacy, the triple (m_1, m_2, l) determines the group and its Nielsen-reduced generating pair. Our main result shows that Γ is free if and only if a certain explicit condition involving m_1, m_2, and the continued-fraction expansion of m_2/m_1 is satisfied. In particular, for fixed (m_1, m_2), the parameter l cleanly separates the free cases from the elliptic ones.

Furthermore, we extend this classification framework to *weighted trees*, where edge lengths are arbitrary positive real numbers. In that context, when the ratio m_2/m_1 is irrational, the possible non-free values of l form a discrete set in $(0, m_1 + m_2)$, and we show these values coincide with the gap lengths appearing in the three-gap theorem [11]. This further highlights a subtle Diophantine phenomenon governing the boundary of freeness for tree-based Schottky groups.

Organization. In Section 2, we recall foundational facts about tree automorphisms and the Nielsen moves used throughout the paper. In Section 3, we define geometric triples and prove our primary classification result, Theorem 3.3. Section 4 extends this classification to weighted trees and identifies a link between axis-intersection lengths and the three-gap theorem via continued fractions, culminating in Theorem 4.8.

2. Preliminaries

In this section we introduces the basic framework of tree automorphisms and sets the geometric foundations used throughout the paper. In particular, we recall standard combinatorial notions such as axes, translation length, and the visual boundary, and describe how hyperbolic automorphisms act by translations along unique bi-infinite geodesics. We then develop the Nielsen reduction process, formulated as elementary moves on generator pairs that strictly decrease the total translation length. This sets up the length-reduction scheme that drives the main theorems, with Lemma 2.7 establishing the key contraction property under Nielsen moves.

2.1. **Trees and Tree Automorphisms.** A *tree* is a connected, acyclic simple graph X with vertex set $\mathcal{V}(X)$ and edge set $\mathcal{E}(X)$. For vertices $v, w \in \mathcal{V}(X)$, the length of the unique simple path $[v, w]$ defines the combinatorial distance

$$d(v, w) = \ell([v, w]). \tag{2.1}$$

An automorphism of X is a graph isomorphism $g : X \to X$. The group of all automorphisms is denoted $\mathrm{Aut}(X)$.

Definition 2.2. An element $g \in \mathrm{Aut}(X)$ is called *elliptic* if it fixes a vertex, *hyperbolic* if it fixes no vertex and no edge (up to inversion), and an *inversion* if it preserves but flips an edge.

We focus on inversion-free actions. The following classical facts (due to Serre [13, Ch. I]) will be used throughout, stated without proof.

Proposition 2.3 (Elliptic case)**.** *If $g \in \mathrm{Aut}(X)$ is elliptic, then it fixes a (possibly degenerate) subtree $T_g \subset X$. Moreover, for any $v \in \mathcal{V}(X)$,*

$$d(v, g.v) = 2d(v, T_g). \tag{2.4}$$

Proposition 2.5 (Hyperbolic case)**.** *If $g \in \mathrm{Aut}(X)$ is hyperbolic, then the minimum displacement*

$$m_g = \min_{v \in \mathcal{V}(X)} d(v, g.v) \in \mathbb{N}_{>0} \tag{2.6}$$

is achieved exactly along a bi-infinite geodesic A_g, called the axis *of g. Moreover,*

$$d(v, g.v) = m_g + 2d(v, A_g). \tag{2.7}$$

Corollary 2.8. *Let $g \in \mathrm{Aut}(X)$ and $v \in \mathcal{V}(X)$. Then*

- *g is elliptic or an inversion if and only if $d(g^{-1}.v, g.v) \leq d(v, g.v)$;*
- *g is hyperbolic if and only if $d(g^{-1}.v, g.v) > d(v, g.v)$.*

Corollary 2.9. *If g is elliptic and u is the midpoint of $[v, g.v]$, then $T_g \cap [v, g.v] = \{u\}$. If g is hyperbolic and $w \in [v, g.v]$ satisfies $d(v, w) = d(v, g.v) - \frac{1}{2}d(g^{-1}.v, g.v)$, then $w \in A_g$ and*

$$m_g = d(g^{-1}.v, g.v) - d(v, g.v). \tag{2.10}$$

Definition 2.11. A subgroup $G < \mathrm{Aut}(X)$ is *discrete* if there is a finite subtree $T \subset X$ such that the pointwise stabilizer

$$G_T = \{g \in G \mid g.v = v \ \forall\, v \in \mathcal{V}(T)\} \tag{2.12}$$

is trivial.

Remark 2.13. If $G < \mathrm{Aut}(X)$ is finitely generated and discrete, then all vertex stabilizers G_v are finite and uniformly bounded in size. Throughout we assume all groups are inversion-free and discrete unless noted.

2.2. **Nielsen Moves and Length Reduction.** Let $\Gamma = \langle \gamma_1, \gamma_2 \rangle < \mathrm{Aut}(X)$ be generated by two hyperbolic automorphisms with translation lengths m_1, m_2. A *Nielsen move* on the ordered pair (γ_1, γ_2) is one of the transformations

$$(\gamma_1, \gamma_2) \mapsto (\gamma_1^{-1}, \gamma_2), \quad (\gamma_1, \gamma_2) \mapsto (\gamma_2, \gamma_1), \quad (\gamma_1, \gamma_2) \mapsto (\gamma_1\gamma_2, \gamma_2), \tag{2.14}$$

or the inverse of one of these. Such moves generate $\mathrm{Aut}(F_2)$ and do not change Γ.

We apply these moves in a translation-length-minimizing fashion. Assign a lexicographic order to length pairs:

$$(m_1, m_2) < (m_1', m_2') \quad \text{if } m_1 < m_1' \text{ or } (m_1 = m_1' \text{ and } m_2 < m_2'). \tag{2.15}$$

A Nielsen move is *length-reducing* if it replaces (γ_1, γ_2) with (γ_1', γ_2') such that

$$(m_1', m_2') \leq_{\mathrm{lex}} (m_1, m_2), \qquad \text{and often } (m_1', m_2') < (m_1, m_2). \tag{2.16}$$

Iterating length-reducing moves produces a finite sequence

$$(\gamma_1, \gamma_2) \to (\gamma_1^{(1)}, \gamma_2^{(1)}) \to \cdots \to (\gamma_1^{(k)}, \gamma_2^{(k)}), \tag{2.17}$$

terminating when no further reduction applies. This process resembles the Euclidean algorithm (see Section 3) and controls the geometry of axes intersections. The geometric consequences of this reduction are developed in Section 3, and form the basis for Theorems 3.3 and 4.8.

Remark 2.18. In contrast to $\mathrm{PSL}(2, \mathbb{R})$, where trace identities guide length minimization, here the reduction is entirely combinatorial and encoded by translation lengths and the quantity $\ell(A_{\gamma_1} \cap A_{\gamma_2})$.

Having established the basic notation and the role of Nielsen transformations in reducing generating pairs, we now turn to the question posed in the Introduction: how the geometric data (m_1, m_2, l) associated to a pair of hyperbolic automorphisms governs the structure of the subgroup $\Gamma = \langle \gamma_1, \gamma_2 \rangle$. In the next section, we develop a Euclidean-style reduction algorithm on geometric triples that leads to a complete classification of when Γ is free of rank two and, in particular, when it admits a Schottky generating pair.

3. Axis-Intersection Criteria and Geometric Triples

In this section we introduce the notion of a geometric triple associated to a pair of hyperbolic automorphisms and develop the length-reduction algorithm that underlies our main classification result. By analyzing successive Nielsen moves and tracing the evolution of translation lengths and intersection data, we reduce any generating pair to one of a few canonical configurations. This procedure culminates in Theorem 3.3, which gives an explicit geometric criterion for determining when such a pair generates a free, discrete subgroup and when it admits a Schottky generating pair. The proof combines combinatorial Nielsen theory with metric bounds derived from axes geometry in the tree.

To present a streamlined formulation of our answer to the question posed in the introduction, we first introduce the following normalization

- Without loss of generality, assume $m_1 \geq m_2$.
- If the axes have at least one common edge, $\mathcal{E}(A_{\gamma_1} \cap A_{\gamma_2}) \neq \varnothing$, let $[v^-, v^+]$ denote this intersection. In addition, we assume that both γ_1 and γ_2 translate from v^- toward v^+ along $[v^-, v^+]$.
- If the axes do not share an edge, let $[v_1, v_2]$ be the unique (possibly degenerate) geodesic path connecting A_{γ_1} and A_{γ_2}, such that $v_1 \in A_{\gamma_1}$ and $v_2 \in A_{\gamma_2}$. We adopt the convention that the axes are *disjoint* in this scenario, even though they may meet at a vertex.

In this setting, the group $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ is described by the translation lengths $m_1 = m_{\gamma_1}$, $m_2 = m_{\gamma_2}$, and the length $l = \ell(A_{\gamma_1} \cap A_{\gamma_2})$ of the intersection of the axes. As we will show below, whether Γ is free of rank 2 is determined by these parameters, closely related to the continued fraction expansion of the ratio $\frac{m_1}{m_2}$.

3.1. **The Main Theorem.** To make our claim precise, we define two sequences inductively. The first is a sequence of positive integers $\{m_i\}_{i=1}^{k+1}$, which is essentially the Euclidean algorithm for m_1 and m_2. Start with m_1, m_2 (where $m_1 \geq m_2 > 0$). For $i \geq 1$, as long as $m_{i+1} > 0$, define:

$$q_i = \left\lfloor \frac{m_i}{m_{i+1}} \right\rfloor, \; m_{i+2} = m_i - q_i m_{i+1}. \tag{3.1}$$

This process terminates at the first index k for which $m_{k+1} = 0$; note that $m_k = \gcd(m_1, m_2)$.

The second is a corresponding sequence of group elements $\{\gamma_i\}_{i=1}^{k+1}$ in Γ. Starting with the given γ_1, γ_2, define for $i \geq 1$:

$$\gamma_{i+2} = \gamma_i \gamma_{i+1}^{-q_i}. \tag{3.2}$$

The sequence $\{\gamma_i\}$ is thus defined for $i = 1, 2, \ldots, k+1$.

We can now state our main classification theorem.

Theorem 3.3. *Under the assumptions above, let*

$$l \coloneqq \ell\Big(A_{\gamma_1} \cap A_{\gamma_2}\Big). \tag{3.4}$$

Then $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ falls into exactly one of the cases (1)-(3) below:

(1) If $l = 0$, then Γ is free of rank two; there is a Schottky pair of generators $(\gamma_{\mathrm{a}}, \gamma_{\mathrm{b}})$ whose axes have disjoint edges.

(2) If $0 < l \leq m_1 + m_2 - \gcd(m_1, m_2)$, there are unique integers j, q with

$$(m_1+m_2) - m_j + (q-1)m_{j+1} < l \leq (m_1+m_2) - m_j + qm_{j+1}, \ 1 \leq j \leq k-1, \ 0 \leq q \leq q_j - 1. \tag{3.5}$$

(a) If $l = (m_1 + m_2) - m_j + qm_{j+1}$, then $\gamma_j\gamma_{j+1}^{-q}$ is hyperbolic; set

$$l_0 := \ell(A_{\gamma_j\gamma_{j+1}^{-q}} \cap \gamma_{j+1}.A_{\gamma_j\gamma_{j+1}^{-q}}). \tag{3.6}$$

i) If $l_0 \geq \frac{m_j-(q+1)m_{j+1}}{2}$ then Γ is not free: it is generated by a hyperbolic element γ_{a} and an elliptic element γ_{b}. The axis $A_{\gamma_{\mathrm{a}}}$ and fixed subtree $T_{\gamma_{\mathrm{b}}}$ are disjoint.

ii) If $l_0 < \frac{m_j-(q+1)m_{j+1}}{2}$ then Γ is free of rank two: there is a Schottky pair $(\gamma_{\mathrm{a}}, \gamma_{\mathrm{b}})$ whose axes $A_{\gamma_{\mathrm{a}}}$ and $A_{\gamma_{\mathrm{b}}}$ are disjoint.

(b) If $l < (m_1 + m_2) - m_j + qm_{j+1}$ then Γ is free of rank two: there is a Schottky pair $(\gamma_{\mathrm{a}}, \gamma_{\mathrm{b}})$ whose axes $A_{\gamma_{\mathrm{a}}}$ and $A_{\gamma_{\mathrm{b}}}$ intersect with v^+ as an endpoint.

(3) If $l > m_1 + m_2 - \gcd(m_1, m_2)$, then Γ is not free: it is generated by a hyperbolic element γ_{a} and an elliptic element γ_{b}. The axis $A_{\gamma_{\mathrm{a}}}$ and fixed subtree $T_{\gamma_{\mathrm{b}}}$ intersect with v^+ as an endpoint are disjoint.

The generating pair $(\gamma_{\mathrm{a}}, \gamma_{\mathrm{b}})$, their translation lengths, and the relevant intersection-length / minimum-distance data are summarized in Table 3.7 below.

Case	Generators $(\gamma_{\mathrm{a}}, \gamma_{\mathrm{b}})$	m_{a}	m_{b}	Relation	$\ell(A_{\gamma_{\mathrm{a}}} \cap A_{\gamma_{\mathrm{b}}})$, $d(A_{\gamma_{\mathrm{a}}}, A_{\gamma_{\mathrm{b}}})$, $\ell(A_{\gamma_{\mathrm{a}}} \cap T_{\gamma_{\mathrm{b}}})$, or $d(A_{\gamma_{\mathrm{a}}}, T_{\gamma_{\mathrm{b}}})$
(1)	(γ_1, γ_2)	m_1	m_2	Disjoint axis-axis	≥ 0
(2a)(i)	$(\gamma_{j+1}, \gamma_j\gamma_{j+1}^{-q-1})$	m_{j+1}	Elliptic	Disjoint axis-tree	$(m_j - (q+1)m_{j+1})/2$
(2a)(ii)			$m_j - (q+1)m_{j+1}$ $-2l_0$	Disjoint axis-axis	l_0
(2b)	$(\gamma_{j+1}, \gamma_j\gamma_{j+1}^{-q})$		$m_j - qm_{j+1}$	Meeting axis-axis	$l - m_1 - m_2$ $+m_j - (q-1)m_{j+1}$
(3)	(γ_k, γ_{k+1})	m_k	Elliptic	Meeting axis-tree	$l - m_1 - m_2 + m_k$

TABLE 3.7. Case summary for $\Gamma = \langle\gamma_1, \gamma_2\rangle$. The entry 'elliptic' in the m_{b} column indicates γ_{b} is elliptic and does not have a translation length.

Remark 3.8 (Tree-Version of the Shimizu Criterion). The axis-overlap threshold stated in Theorem 3.3 refines a classical qualitative principle: that two hyperbolic automorphisms of a tree generate a free group provided their axes do not overlap "too much." A precise formulation of this principle appears as a consequence of Tits' ping-pong arguments in Serre's exposition [13]: if $g, h \in \mathrm{Aut}(X)$ are hyperbolic and

$$\ell(A_g \cap A_h) < \min\{\ell(g), \ell(h)\}, \tag{3.9}$$

then $\langle g, h\rangle$ is a free rank-two subgroup of $\mathrm{Aut}(X)$. This may be viewed as a tree-theoretic analog of the classical axis-separation criteria for Schottky-type subgroups in Kleinian group theory (often discussed under the heading of a Shimizu lemma, not to be confused with

Shimizu's inequality for parabolic elements). Our Theorem 3.3 sharpens this picture in the simplicial setting: once the continued-fraction reduction $\{\gamma_i\}$ of (3.2) is taken into account, the critical threshold for the original pair (γ_1, γ_2) turns out to be the precise value

$$\ell(A_{\gamma_1} \cap A_{\gamma_2}) = m_1 + m_2 - \gcd(m_1, m_2), \tag{3.10}$$

with elliptic behavior forced exactly when this bound is exceeded. Thus, the present result furnishes a quantitative completion of the classical axis-overlap criterion for rank-two free subgroups in $\mathrm{Aut}(X)$.

3.2. **Geometric Triples for Generating Pairs.** Before proving the theorem, we establish some further notions for a clearer description.

Definition 3.11. Let X be a tree and let $\gamma_1, \gamma_2 \in \mathrm{Aut}(X)$ be hyperbolic automorphisms that generate a discrete subgroup. Let $m_1, m_2 > 0$ be their translation lengths, and let $l = \ell(A_{\gamma_1} \cap A_{\gamma_2})$ be the length of the intersection of their axes. The triple (l, m_1, m_2) is called the *geometric triple* of the generating pair (γ_1, γ_2).

If we assume $m_1 \geq m_2$, the definition guarantees that the geometric triple (l, m_1, m_2) of a hyperbolic generating pair satisfies exactly one of the following three conditions:

(I) $l \geq m_2$.

(II) $0 < l < m_2$.

(III) $l = 0$.

Having established the possible cases for the geometric triple, we now prove that the group $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ is automatically Schottky in two of the scenarios.

Proposition 3.12 (cf. [1], Proposition 3.4)**.** *Let (γ_1, γ_2) be a pair of hyperbolic automorphisms with translation lengths $m_1 \geq m_2$, and suppose its geometric triple (l, m_1, m_2) falls in either case (II) or (III). Then, $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ is free and discrete, and (γ_1, γ_2) is a pair of Schottky generators.*

Proof. We proceed by constructing explicit fundamental domains for the action of Γ on X.

Case (II): $0 < l < m_2$. Recall that we denoted the intersection of the axes by $[v^-, v^+]$, and assumed without loss of generality that both γ_1 and γ_2 translate from v^- toward v^+. We denote four edges as follows (see Fig. 3.13):

- For $i = 1, 2$, let e_i^- be the edge in $A_{\gamma_i} \setminus [v^-, v^+]$ that is adjacent to v^-.

- For $i = 1, 2$, define $e_i^+ = \gamma_i . e_i^-$.

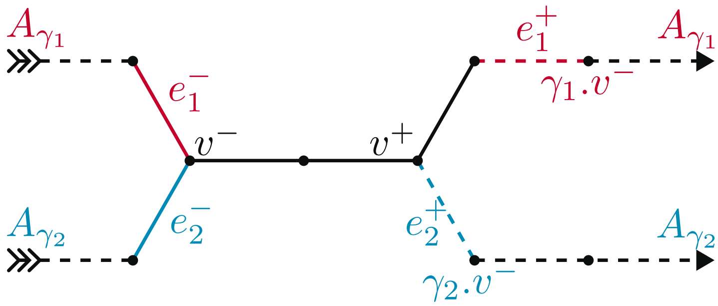

FIGURE 3.13. The fundamental domain in case (II), where we set $l = 2$, $m_1 = 3$, and $m_2 = 5$. Irrelevant vertices and edges are omitted. Edges not in the fundamental domain are dashed.

Case (III): $l = 0$. In this scenario we denoted the unique (possibly degenerate) path connecting the two axes by $[v_1, v_2]$, where $v_i \in \mathcal{V}(A_{\gamma_i})$ for $i = 1, 2$. We denote the edges differently (see Fig. 3.14):

- For $i = 1, 2$, let e_i^- be the edge in $[\gamma_i^{-1}.v_i, v_i]$ adjacent to v_i.
- For $i = 1, 2$, define $e_i^+ = \gamma_i.e_i^-$.

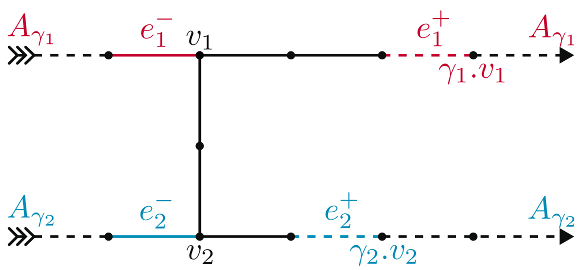

FIGURE 3.14. The fundamental domain in case (III), where we set $m_1 = 2$, and $m_2 = 3$.

In either case, the automorphism γ_i translates e_i^- to e_i^+, and the four edges e_i^+ and e_i^- are disjoint from the edge sets $\mathcal{E}([v^+, v^-])$ or $\mathcal{E}([v_1, v_2])$. Define H_i^+ (and H_i^-) for $i = 1, 2$ to be the connected component of $X \setminus \{e_i^+\}$ ($X \setminus \{e_i^-\}$, respectively) that does not meet the specified path $[v^+, v^-]$ or $[v_1, v_2]$. Then the subtrees $H_i^\pm$ are non-empty and pairwise disjoint, satisfying the Ping-Pong condition:

$$\gamma_i.(X \setminus H_i^-) = e_i^+ \cup H_i^+, \quad \gamma_i^{-1}.(X \setminus H_i^+) = e_i^- \cup H_i^-, \quad i = 1, 2. \tag{3.15}$$

Let $\mathcal{Y}$ be the connected set of vertices and edges, determined by its edge set

$$\mathcal{E}(\mathcal{Y}) = \mathcal{E}(X) \setminus (\mathcal{E}(H_1^+ \cup H_1^- \cup H_2^+ \cup H_2^-) \cup \{e_1^+, e_2^+\}), \tag{3.16}$$

and maximal subtree $Y_0 \subset \mathcal{Y}$, where

$$\mathcal{E}(Y_0) = \mathcal{E}(X) \setminus (\mathcal{E}(H_1^+ \cup H_1^- \cup H_2^+ \cup H_2^-) \cup \{e_1^+, e_2^+, e_1^-, e_2^-\}). \tag{3.17}$$

By the Ping-Pong Lemma, $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ is a Schottky group with respect to generators γ_1 and γ_2 and the fundamental domain $\mathcal{Y}$. The group Γ is thus discrete and free of rank 2.

$\square$

If the geometric triple (l, m_1, m_2) is in case (I), we apply a *Nielsen transformation* to replace (γ_1, γ_2) by a new generating pair $(\gamma_1\gamma_2^{-1}, \gamma_2)$, and consider an associated triple. Define the candidate triple algebraically as:

$$(l', m_1', m_2) = (l - m_2, m_1 - m_2, m_2). \tag{3.18}$$

This candidate triple (l', m_1', m_2) necessarily satisfies one of the following conditions:

(I) $l' \geq \min(m_1', m_2)$ and $m_1' > 0$,

(II) $l' < \min(m_1', m_2)$ and $l' > 0$,

(III) $m_1' > 0$ and $l' = 0$, or

(IV) $m_1' = 0$.

We note that the first three cases are analogous to our original classification, possibly after swapping m_1' with m_2. The case (IV) is new.

Under specific conditions, this candidate triple indeed realizes itself as the geometric triple for the new generating pair.

Proposition 3.19 (cf. [1], Proposition 3.5, Case (2)(ii))**.** *Suppose the geometric triple (l, m_1, m_2) is in case (I), and its associated candidate triple (l', m_1', m_2) is in case (I) or (II). Then, the element $\gamma_1\gamma_2^{-1}$ is hyperbolic. In addition, the candidate triple (l', m_1', m_2) is the geometric triple of the new generating pair $(\gamma_1\gamma_2^{-1}, \gamma_2)$, with $v^+ \in \mathcal{V}(A_{\gamma_2} \cap A_{\gamma_1\gamma_2^{-1}})$ as an endpoint.*

Proof. First we establish the hyperbolicity of $\gamma_1\gamma_2^{-1}$ with the predicted translation length. Refer to Fig. 3.21, the inclusion $v^+ \in \mathcal{V}(A_{\gamma_1} \cap A_{\gamma_2})$ implies that $\gamma_1^{-1}.v^+ \in \mathcal{V}(A_{\gamma_1})$; by the assumption $l' = l - m_2 > 0$, $\gamma_2^{-1}.v^+ \in \mathcal{V}(A_{\gamma_1})$ as well. Since γ_1 and γ_2 translate toward the same direction, the distance between the latter two points is

$$d(\gamma_1^{-1}.v^+, \gamma_2^{-1}.v^+) = d(v^+, \gamma_1^{-1}.v^+) - d(v^+, \gamma_2^{-1}.v^+) = m_1 - m_2. \tag{3.20}$$

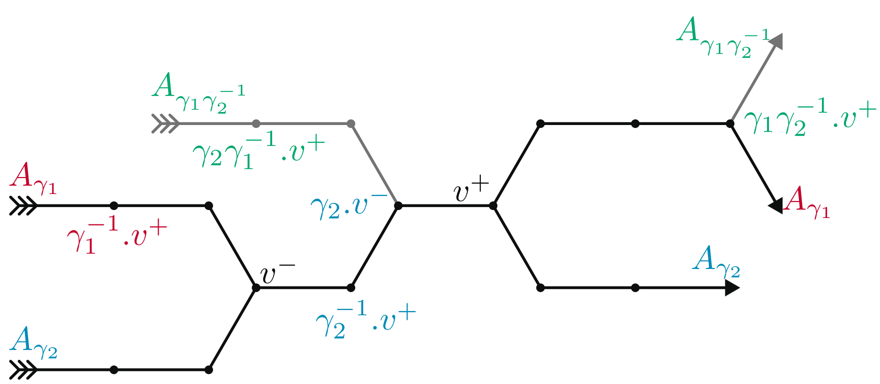

FIGURE 3.21. Generator reduction from the geometric triple $(l, m_1, m_2) = (3, 5, 2)$ to $(l', m_1', m_2) = (1, 3, 2)$. This figure describes a reduction to case (II); a figure for case (I) is analogous and is left to the reader.

Denote the path $P = [\gamma_1^{-1}.v^+, \gamma_2^{-1}.v^+]$ and apply the automorphisms γ_1 and γ_2 to it:

- Since $P \subset A_{\gamma_1}$, the automorphism γ_1 translates P to a path $[v^+, \gamma_1\gamma_2^{-1}.v^+]$. The length of this translated path is again $m_1 - m_2$, and the path emanates from v^+ to $\gamma_1\gamma_2^{-1}.v^+$ in the positive direction along A_{γ_1}.
- Since $l - m_2 > 0$ is strict, the edge in $\mathcal{E}(P)$ adjacent to $\gamma_2^{-1}.v^+$ is contained in A_{γ_2}. The automorphism γ_2 translates P to a path $[v^+, \gamma_2\gamma_1^{-1}.v^+]$. The length of this translated path is also $m_1 - m_2$, while it emanates from v^+ in the negative direction along $A_{\gamma_1} \cap A_{\gamma_2}$.

This configuration implies that the points $\gamma_1\gamma_2^{-1}.v^+$ and $\gamma_2\gamma_1^{-1}.v^+$ lie on different directions initiating from v^+, specifically:

$$d(\gamma_1\gamma_2^{-1}.v^+, \gamma_2\gamma_1^{-1}.v^+) = 2(m_1 - m_2) > m_1 - m_2 = d(v^+, \gamma_1\gamma_2^{-1}.v^+). \tag{3.22}$$

By Corollaries 2.8 and 2.9, this inequality confirms that $\gamma_1\gamma_2^{-1}$ is hyperbolic with translation length $m_1 - m_2$.

Next, we derive the two endpoints of the intersection $A_{\gamma_2} \cap A_{\gamma_1\gamma_2^{-1}}$ to confirm the geometric triple and the inclusion $v^+ \in \mathcal{V}(A_{\gamma_1\gamma_2^{-1}})$. We know from our first argument that the path $[v^+, \gamma_1\gamma_2^{-1}.v^+] \subset A_{\gamma_1\gamma_2^{-1}}$ and $[v^+, \gamma_1\gamma_2^{-1}.v^+] \cap A_{\gamma_2} = \{v^+\}$, establishing that v^+ is one endpoint of the intersection $A_{\gamma_2} \cap A_{\gamma_1\gamma_2^{-1}}$. To obtain the other endpoint, let $j \geq 0$ be the smallest natural number such that $j(m_1 - m_2) > l - m_1$. This condition, as illustrated in Fig. 3.21, ensures that

$$\gamma_1^{-1}(\gamma_2\gamma_1^{-1})^j.v^+ \in \mathcal{V}(A_{\gamma_1} \setminus A_{\gamma_2}), \quad \text{while} \quad \gamma_1^{-1}(\gamma_2\gamma_1^{-1})^{j-1}.v^+ \in \mathcal{V}(A_{\gamma_1} \cap A_{\gamma_2}). \tag{3.23}$$

(Note that for $j = 0$, we interpret $\gamma_1^{-1}(\gamma_2\gamma_1^{-1})^{-1}.v^+$ as $\gamma_2^{-1}.v^+$, which lies in $A_{\gamma_1} \cap A_{\gamma_2}$.)

Recall that v^- is the other endpoint of $A_{\gamma_1} \cap A_{\gamma_2}$. The inclusions above imply that

$$[\gamma_1^{-1}(\gamma_2\gamma_1^{-1})^j.v^+, \gamma_1^{-1}(\gamma_2\gamma_1^{-1})^{j-1}.v^+] \cap A_{\gamma_2} = [v^-, \gamma_1^{-1}(\gamma_2\gamma_1^{-1})^{j-1}.v^+]. \tag{3.24}$$

Translating this configuration by γ_2 yields:

$$[(\gamma_2\gamma_1^{-1})^{j+1}.v^+, (\gamma_2\gamma_1^{-1})^j.v^+] \cap A_{\gamma_2} = [\gamma_2.v^-, (\gamma_2\gamma_1^{-1})^j.v^+]. \tag{3.25}$$

Note that $[(\gamma_2\gamma_1^{-1})^{j+1}.v^+, (\gamma_2\gamma_1^{-1})^j.v^+] \subset A_{\gamma_1\gamma_2^{-1}}$; the result above identifies $\gamma_2.v^-$ as the other endpoint of $A_{\gamma_2} \cap A_{\gamma_1\gamma_2^{-1}}$. Then, the length of this intersection is computed as

$$\ell(A_{\gamma_2} \cap A_{\gamma_1\gamma_2^{-1}}) = d(\gamma_2.v^-, v^+) = d(v^-, v^+) - d(v^-, \gamma_2.v^-) = l - m_2 = l'. \tag{3.26}$$

This concludes our claim that the candidate triple (l', m_1', m_2) is indeed the geometric triple for the generating pair $(\gamma_1\gamma_2^{-1}, \gamma_2)$.

□

Proposition 3.27 (cf. [1], Proposition 3.5, Case (2)(iii))**.** *Suppose the geometric triple (l, m_1, m_2) is in case (I) and its associated candidate triple (l', m_1', m_2) is in case (III): $l' = 0$ and $m_1' > 0$. Let $l_0 = \ell(A_{\gamma_1} \cap \gamma_2.A_{\gamma_1})$. Then:*

- *If $l_0 \geq \frac{m_1 - m_2}{2}$, then $\gamma_1\gamma_2^{-1}$ is elliptic. Furthermore, the axis A_{γ_2} and the fixed tree $T_{\gamma_1\gamma_2^{-1}}$ are disjoint, with $d(A_{\gamma_2}, T_{\gamma_1\gamma_2^{-1}}) = \frac{m_1 - m_2}{2}$.*

- *If* $l_0 < \frac{m_1-m_2}{2}$*, then* $\gamma_1\gamma_2^{-1}$ *is hyperbolic. Furthermore, the geometric triple of* Γ *with respect to the new generating pair* $(\gamma_1\gamma_2^{-1}, \gamma_2)$ *is* $(0, m_1 - m_2 - 2l_0, m_2)$*, with the minimum distance* $d(A_{\gamma_2}, A_{\gamma_1\gamma_2^{-1}}) = l_0$*.*

Proof. As established in the proof of Proposition 3.19, the path $\gamma_1.P = [v^+, \gamma_1\gamma_2^{-1}.v^+]$ has length $m_1 - m_2$ and emanates from v^+ to $\gamma_1\gamma_2^{-1}.v^+$ in the positive direction along A_{γ_1}. The length of $\gamma_2.P = [v^+, \gamma_2\gamma_1^{-1}.v^+]$ is also $m_1 - m_2$, while the condition $l = m_2$ implies that $\gamma_2.P$ does not follow the negative direction along $A_{\gamma_1} \cap A_{\gamma_2}$ when emanating from v^+. Therefore, $[v^+, \gamma_2\gamma_1^{-1}.v^+]$ either emanates from v^+ in a direction different from those in $A_{\gamma_1} \cup A_{\gamma_2}$, or shares common edges with $[v^+, \gamma_1\gamma_2^{-1}.v^+]$.

Notice that $\gamma_2\gamma_1^{-1}.v^+ \in \mathcal{V}(\gamma_2.A_{\gamma_1})$, while the condition $l = m_2$ implies that $\gamma_1\gamma_2^{-1}.v^+ = \gamma_1.v^- \in \mathcal{V}(A_{\gamma_1})$. On the other hand, $\gamma_2.v^+ \in \mathcal{V}(\gamma_2.A_{\gamma_1})$ and $[v^+, \gamma_2.v^+] \subset A_{\gamma_2} \setminus A_{\gamma_1}$, hence v^+ is an endpoint of the intersection $A_{\gamma_1} \cap \gamma_2.A_{\gamma_1}$. It follows that

$$\ell(\gamma_1.P \cap \gamma_2.P) = \min(l_0, m_1 - m_2), \tag{3.28}$$

and

$$d(\gamma_1\gamma_2^{-1}.v^+, \gamma_2\gamma_1^{-1}.v^+) = 2(m_1 - m_2 - \ell(\gamma_1.P \cap \gamma_2.P)) = 2\max(0, m_1 - m_2 - l_0). \tag{3.29}$$

We continue by discussing the two cases claimed in the proposition.

Case (i): $l_0 \geq \frac{m_1-m_2}{2}$**.** In this case, the distance

$$d(\gamma_1\gamma_2^{-1}.v^+, \gamma_2\gamma_1^{-1}.v^+) = 2\max(0, m_1 - m_2 - l_0) \leq m_1 - m_2 = d(v^+, \gamma_1\gamma_2^{-1}.v^+), \tag{3.30}$$

thus Corollary 2.8 implies that $\gamma_1\gamma_2^{-1}$ is elliptic.

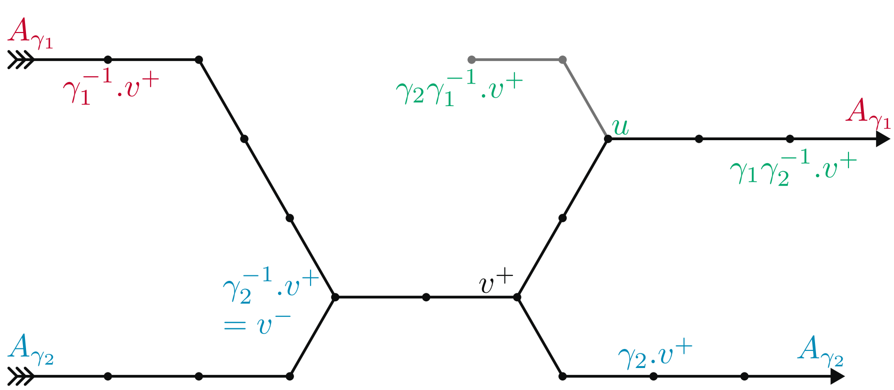

FIGURE 3.31. An elliptic reduction: the geometric triple $(l, m_1, m_2) = (2, 6, 2)$ yields a candidate triple in case (III) with $l_0 = 2$. The element $\gamma_1\gamma_2^{-1}$ is elliptic, and a vertex u fixed by it is marked.

Corollary 2.9 further implies that $[v^+, \gamma_1\gamma_2^{-1}.v^+]$ intersects with $T_{\gamma_1\gamma_2^{-1}}$ at the midpoint u, thus

$$d(A_{\gamma_2}, T_{\gamma_1\gamma_2^{-1}}) = d(v^+, u) = \frac{1}{2}d(v^+, \gamma_1\gamma_2^{-1}.v^+) = \frac{m_1 - m_2}{2}. \tag{3.32}$$

Case (ii): $l_0 < \frac{m_1 - m_2}{2}$. First, we establish the hyperbolicity: the distance

$$d(\gamma_1\gamma_2^{-1}.v^+, \gamma_2\gamma_1^{-1}.v^+) = 2(m_1 - m_2 - l_0) > m_1 - m_2 = d(v^+, \gamma_1\gamma_2^{-1}.v^+), \tag{3.33}$$

thus Corollary 2.8 implies that $\gamma_1\gamma_2^{-1}$ is hyperbolic, and Corollary 2.9 implies that the translation length

$$m_{\gamma_1\gamma_2^{-1}} = d(\gamma_1\gamma_2^{-1}.v^+, \gamma_2\gamma_1^{-1}.v^+) - d(v^+, \gamma_1\gamma_2^{-1}.v^+) = m_1 - m_2 - 2l_0. \tag{3.34}$$

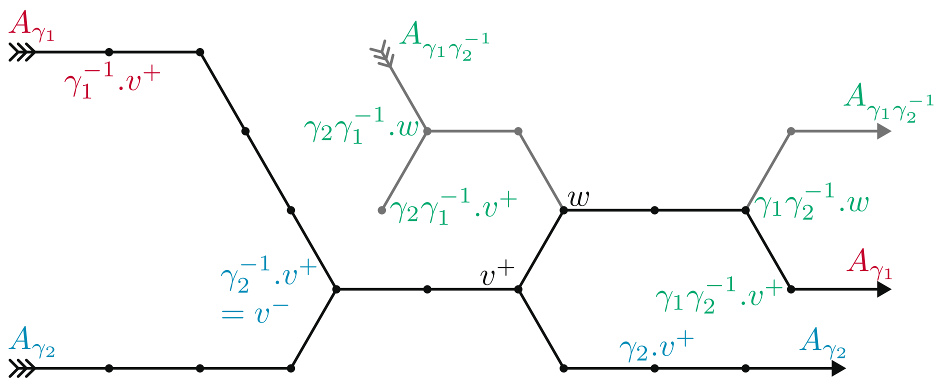

FIGURE 3.35. A hyperbolic reduction: the geometric triple $(l, m_1, m_2) = (2, 6, 2)$ yields a candidate triple in case (III) with $l_0 = 1$. The new element $\gamma_1\gamma_2^{-1}$ is hyperbolic, and its axis is disjoint from A_{γ_2}.

We now determine the geometric triple by considering the positional relation between A_{γ_2} and $A_{\gamma_1\gamma_2^{-1}}$. Let w be the closest point projection of v^+ to $A_{\gamma_1\gamma_2^{-1}}$. By Corollary 2.9,

$$d(v^+, w) = d(v^+, \gamma_1\gamma_2^{-1}.v^+) - \frac{1}{2}d(\gamma_1\gamma_2^{-1}.v^+, \gamma_2\gamma_1^{-1}.v^+) = l_0. \tag{3.36}$$

This shows our claim if $l_0 > 0$. Otherwise, the condition $l_0 = 0$ implies that $w = v^+$, and the property of hyperbolic automorphisms implies that

$$[v^+, \gamma_1\gamma_2^{-1}.v^+] \cup [v^+, \gamma_2\gamma_1^{-1}.v^+] \subset A_{\gamma_1\gamma_2^{-1}}. \tag{3.37}$$

As we have shown, neither $[v^+, \gamma_1\gamma_2^{-1}.v^+]$ nor $[v^+, \gamma_2\gamma_1^{-1}.v^+]$ emanates from v^+ along A_{γ_2}. Therefore $A_{\gamma_2} \cap A_{\gamma_1\gamma_2^{-1}} = \{v^+\}$, which also yields our desired conclusion.

□

Proposition 3.38. *Suppose the geometric triple (l, m_1, m_2) is in case (I) and its associated candidate triple (l', m_1', m_2) is in case (IV): $m_1' = 0$. Then $\gamma_1\gamma_2^{-1}$ is elliptic. Furthermore, the intersection $A_{\gamma_2} \cap T_{\gamma_1\gamma_2^{-1}}$ has length l', with v^+ as an endpoint.*

Proof. We first derive the ellipticity of $\gamma_1\gamma_2^{-1}$. The condition $m_1' = 0$ implies $m_1 = m_2$. Since the original triple (l, m_1, m_2) is in case (I), the resulting relation $l \geq m_1 = m_2$ implies that

$$\gamma_1^{-1}.v^+, \gamma_2^{-1}.v^+ \in \mathcal{V}(A_{\gamma_1} \cap A_{\gamma_2}),\ d(\gamma_1^{-1}.v^+, v^+) = m_1 = d(\gamma_2^{-1}.v^+, v^+). \tag{3.39}$$

Since γ_1 and γ_2 translate toward the same direction, $\gamma_1^{-1}.v^+ = \gamma_2^{-1}.v^+$, which implies $\gamma_1\gamma_2^{-1}.v^+ = v^+$; see Fig. 3.40.

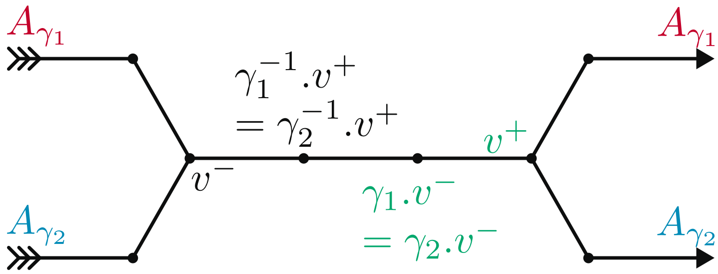

FIGURE 3.40. An elliptic reduction: the geometric triple $(l, m_1, m_2) = (3, 2, 2)$ yields a candidate triple in case (IV). The new element $\gamma_1\gamma_2^{-1}$ is elliptic, and the path $[\gamma_1.v^-, v^+]$ is fixed.

This fixed point demonstrates the ellipticity of $\gamma_1\gamma_2^{-1}$.

Next, we obtain the two endpoints of $A_{\gamma_2} \cap T_{\gamma_1\gamma_2^{-1}}$ to confirm the intersection length. Let e^+ be the edge in $\mathcal{E}(A_{\gamma_2} \setminus A_{\gamma_1})$ that is incident with v^+. Since $\gamma_1^{-1}.e^+ \notin \mathcal{E}(A_{\gamma_2})$ and $\gamma_2^{-1}.e^+ \in \mathcal{E}(A_{\gamma_2})$, $\gamma_1^{-1}.e^+ \neq \gamma_2^{-1}.e^+$, which further implies $e^+ \neq \gamma_1\gamma_2^{-1}.e^+$. As we have shown earlier, e^+ and $\gamma_1\gamma_2^{-1}.e^+$ share the vertex v^+, confirming that v^+ is one endpoint of $A_{\gamma_2} \cap T_{\gamma_1\gamma_2^{-1}}$.

For a similar reason, if we let e^- be the edge in $\mathcal{E}(A_{\gamma_2} \setminus A_{\gamma_1})$ incident with v^-, then

$$\gamma_1.e^- \neq \gamma_2.e^-, \quad \gamma_1.e^- \cap \gamma_2.e^- = \gamma_2.v^-. \tag{3.41}$$

Writing $\gamma_1.e^-$ as $(\gamma_1\gamma_2^{-1}).(\gamma_2.e^-)$, we observe that $\gamma_2.v^-$ is the other endpoint of $A_{\gamma_2} \cap T_{\gamma_1\gamma_2^{-1}}$. Consequently,

$$\ell(A_{\gamma_2} \cap T_{\gamma_1\gamma_2^{-1}}) = d(v^+, \gamma_2.v^-) = d(v^+, v^-) - d(v^-, \gamma_2.v^-) = l - m_2 = l'. \tag{3.42}$$

□

3.3. **Proof of Theorem 3.3.** Let (γ_1, γ_2) be a pair of hyperbolic automorphisms generating a discrete subgroup $\langle\gamma_1, \gamma_2\rangle$, and (l, m_1, m_2) be the associated geometric triple. We apply Nielsen transformations to the generating pair, and reduce the geometric triple simultaneously, following the rule (cf. [1], Algorithm 4.1):

- If $m_1 < m_2$, replace (γ_1, γ_2) with (γ_2, γ_1), and (l, m_1, m_2) with (l, m_2, m_1).
- If $m_1 \geq m_2$ and the triple satisfies case (I), replace (γ_1, γ_2) with $(\gamma_1\gamma_2^{-1}, \gamma_2)$, and (l, m_1, m_2) with $(l - m_2, m_1 - m_2, m_2)$.
- Terminate the reduction procedure if $m_1 \geq m_2$ and the triple satisfies case (II), (III), or (IV).

The reduction resembles the Euclidean division algorithm and thus always terminates.

Lemma 3.43. *Define the sequences $\{m_i\}_{i=1}^{k+1}$ and $\{q_i\}_{i=1}^{k-1}$ as in the beginning of Section 3. Starting with the triple (l, m_1, m_2), $m_1 \geq m_2$, the change of the two translation lengths in the* *triple agrees with the Euclidean algorithm before termination:*

$$\begin{aligned}
&(m_1, m_2) \to (m_1 - m_2, m_2) \to \cdots \to (m_1 - q_1 m_2, m_2) = (m_3, m_2) \\
&\to (m_2, m_3) \to (m_2 - m_3, m_3) \to \cdots \to (m_2 - q_2 m_3, m_3) = (m_4, m_3) \\
&\qquad \vdots \\
&\to (m_{k-1}, m_k) \to (m_{k-1} - m_k, m_k) \to \cdots \to (m_{k-1} - q_{k-1} m_k, m_k) = (m_{k+1}, m_k) \\
&\to (m_k, m_{k+1}) = (\gcd(m_1, m_2), 0).
\end{aligned} \tag{3.44}$$

Denote the sequence of triples occurring in this algorithm by $\{(l^{(n)}, m_1^{(n)}, m_2^{(n)})\}_{n=0}^N$*, so* $(l^{(0)}, m_1^{(0)}, m_2^{(0)}) = (l, m_1, m_2)$*. Then for any state index* $n \geq 0$*, the triple* $(l^{(n)}, m_1^{(n)}, m_2^{(n)})$ *satisfies*

$$l^{(n)} - m_1^{(n)} - m_2^{(n)} = l - m_1 - m_2. \tag{3.45}$$

Furthermore, the algorithm terminates if and only if one of the following occurs:

- *The algorithm terminates in case (II) or (III) if* $l^{(n)} < m_2^{(n)}$ *for a certain index* n*. Specifically, it terminates in case (III) if* $l^{(n)} = 0$*; otherwise, it terminates in case (II).*
- *If the condition* $l^{(n)} \geq m_2^{(n)}$ *persists until the final state* $(l^{(N)}, m_1^{(N)}, m_2^{(N)}) = (l - m_1 - m_2 + \gcd(m_1, m_2), \gcd(m_1, m_2), 0)$ *occurs, the algorithm terminates in case (IV).*

The lemma is clear regarding the description of the algorithm and the four cases for the geometric triple. We now take a closer look at the algorithm based on this lemma:

Proposition 3.46. *Let* (l, m_1, m_2) *be a geometric triple with* $m_1 \geq m_2$*, and define the sequences* $\{m_i\}_{i=1}^{k+1}$ *and* $\{q_i\}_{i=1}^{k-1}$ *as in the beginning of Section 3. The following cases exhaust the possible outcomes of the reduction algorithm:*

- *If* $l = 0$*, the algorithm terminates in case (III) without any reduction steps.*
- *If* $0 < l < m_2$*, the algorithm terminates in case (II) without any reduction steps.*
- *If* $m_2 < l < m_1 + m_2 - \gcd(m_1, m_2)$*, there are unique integers* j, q *with*
$$(m_1 + m_2) - m_j + (q-1)m_{j+1} < l \leq (m_1 + m_2) - m_j + q m_{j+1},\ 1 \leq j \leq k-1,\ 0 \leq q \leq q_j - 1. \tag{3.47}$$
  - *If* $l = (m_1 + m_2) - m_j + q m_{j+1}$*, the algorithm terminates in case (III) after reduction.*
  - *If* $l < (m_1 + m_2) - m_j + q m_{j+1}$*, the algorithm terminates in case (II) after reduction.*
- *If* $l \geq (m_1 + m_2) - \gcd(m_1, m_2)$*, the algorithm terminates in case (IV) after reduction.*

Proof. Before giving the proof, we observe that the difference $l^{(n)} - \min(m_1^{(n)}, m_2^{(n)})$ is non-increasing. The algorithm continues whenever $l^{(n)} - \min(m_1^{(n)}, m_2^{(n)})$ is non-negative, and terminates when it becomes negative, possibly after a final swap of $m_1^{(n)}$ and $m_2^{(n)}$.

The proof for the first two outcomes is clear. If $l \geq m_1 + m_2 - \gcd(m_1, m_2)$, then before the last step of the Euclidean algorithm, we have translation lengths

$$m_1^{(N-1)} = m_2^{(N-1)} = \gcd(m_1, m_2). \tag{3.48}$$

By Lemma 3.43,

$$l^{(N-1)} = l - m_1 - m_2 + 2\gcd(m_1, m_2) \geq \gcd(m_1, m_2). \tag{3.49}$$

This inequality together with our observation implies that the algorithm continues until the end of the Euclidean division algorithm for (m_1, m_2). By Lemma 3.43, this corresponds to a termination in case (IV).

Suppose now the following inequality holds for integers $1 \leq j \leq k-1$ and $0 \leq q \leq q_j - 1$:

$$(m_1 + m_2) - m_j + (q-1)m_{j+1} < l \leq (m_1 + m_2) - m_j + qm_{j+1}. \tag{3.50}$$

As in Lemma 3.43, we consider two consecutive states for the translation lengths in the algorithm:

$$(m_1^{(n)}, m_2^{(n)}) = (m_j - qm_{j+1}, m_{j+1}), \quad (m_1^{(n+1)}, m_2^{(n+1)}) = (m_j - (q+1)m_{j+1}, m_{j+1}). \tag{3.51}$$

The corresponding quantity $l^{(n)}$ is

$$l^{(n)} = l - m_1 - m_2 + m_1^{(n)} + m_2^{(n)} = l - m_1 - m_2 + m_j - (q-1)m_{j+1}. \tag{3.52}$$

The inequality for l implies $0 < l^{(n)} \leq m_{j+1}$. Together with our observation, this relation implies one of the followings:

- If $0 < l^{(n)} < m_{j+1}$, 3.43 implies that the triple (l, m_1, m_2) terminates in case (II) at the n-th state.
- If $l^{(n)} = m_{j+1}$, the triple (l, m_1, m_2) terminates in case (III) at the $(n+1)$-th state.

□

Proof of Theorem 3.3. Perform the algorithm for the generating pair (γ_1, γ_2) and the associated geometric triple (l, m_1, m_2) as described at the beginning of this subsection. Parallel to the transformation of (m_1, m_2) described in Lemma 3.43, the generating pair transforms as follows:

$$\begin{aligned}
&(\gamma_1, \gamma_2) \to (\gamma_1\gamma_2^{-1}, \gamma_2) \to \cdots \to (\gamma_1\gamma_2^{-q_1}, \gamma_2) = (\gamma_3, \gamma_2) \\
&\to (\gamma_2, \gamma_3) \to (\gamma_2\gamma_3^{-1}, \gamma_3) \to \cdots \to (\gamma_2\gamma_3^{-q_2}, \gamma_3) = (\gamma_4, \gamma_3) \\
&\qquad\vdots \\
&\to (\gamma_{k-1}, \gamma_k) \to (\gamma_{k-1}\gamma_k^{-1}, \gamma_k) \to \cdots \to (\gamma_{k-1}\gamma_k^{-q_{k-1}}, \gamma_k) = (\gamma_{k+1}, \gamma_k) \\
&\to (\gamma_k, \gamma_{k+1}).
\end{aligned} \tag{3.53}$$

We similarly denote this sequence of generating pairs by $(\gamma_1^{(n)}, \gamma_2^{(n)})$. By Proposition 3.19, the candidate triple $(l^{(n)}, m_1^{(n)}, m_2^{(n)})$ is the geometric triple for the generating pair $(\gamma_1^{(n)}, \gamma_2^{(n)})$, except for the final step that reduces the triple to cases (III) or (IV). We will discuss the outcome of the algorithm as described in Proposition 3.46.

Termination in case (II) or (III) without reduction. If the algorithm terminates without reduction steps, then $l < m_2$. Proposition 3.12 proves that Γ is free and that (γ_1, γ_2) is a pair of Schottky generators. When $l = 0$, this corresponds to case (1) in Theorem 3.3. When $0 < l < m_2$, it is straightforward to check that the pair corresponds to a specific instance of case (2b) with $j = 1$ and $q = 0$.

Termination in case (II) after reduction. If the reduction terminates at a candidate triple in case (II), Proposition 3.46 shows that l satisfies the inequality

$$(m_1 + m_2) - m_j + (q-1)m_{j+1} < l < (m_1 + m_2) - m_j + qm_{j+1}, \tag{3.54}$$

for certain $1 \leq j \leq k-1$ and $0 \leq q \leq q_j - 1$, except for $(j,q) = (1,0)$ discussed earlier. Proposition 3.46 further implies that the algorithm terminates at the geometric triple

$$(l^{(n)}, m_1^{(n)}, m_2^{(n)}) = (l - m_1 - m_2 + m_j - (q-1)m_{j+1}, m_j - qm_{j+1}, m_{j+1}) \tag{3.55}$$

of the generating pair $(\gamma_\mathrm{b}, \gamma_\mathrm{a}) = (\gamma_1^{(n)}, \gamma_2^{(n)}) = (\gamma_j\gamma_{j+1}^{-q}, \gamma_{j+1})$. The Schottky property for the generating pair is guaranteed by Proposition 3.12, and the translation lengths m_a, m_b and the intersection length $\ell(A_{\gamma_\mathrm{a}} \cap A_{\gamma_\mathrm{b}})$ are derived from the geometric triple $(l^{(n)}, m_1^{(n)}, m_2^{(n)})$. Proposition 3.19 also guarantees that v^+ remains as an endpoint of the axes intersection after each reduction step, hence v^+ is an endpoint of $A_{\gamma_\mathrm{a}} \cap A_{\gamma_\mathrm{b}}$. This corresponds to case (2b) in Theorem 3.3.

Termination in case (III) after reduction. If the reduction terminates at a candidate triple in case (III), Proposition 3.46 shows that

$$l = (m_1 + m_2) - m_j + qm_{j+1} \tag{3.56}$$

for certain $1 \leq j \leq k-1$ and $0 \leq q \leq q_j - 1$, except for $(j,q) = (k-1, q_{k-1}-1)$. The algorithm terminates at the candidate triple

$$(l^{(n+1)}, m_1^{(n+1)}, m_2^{(n+1)}) = (0, m_j - (q+1)m_{j+1}, m_{j+1}) \tag{3.57}$$

of the generating pair $(\gamma_\mathrm{b}, \gamma_\mathrm{a}) = (\gamma_1^{(n+1)}, \gamma_2^{(n+1)}) = (\gamma_j\gamma_{j+1}^{-q-1}, \gamma_{j+1})$.

From the description of the algorithm, the preceding generating pair is $(\gamma_1^{(n)}, \gamma_2^{(n)}) = (\gamma_j\gamma_{j+1}^{-q}, \gamma_{j+1})$, with the geometric triple $(l^{(n)}, m_1^{(n)}, m_2^{(n)}) = (m_{j+1}, m_j - qm_{j+1}, m_{j+1})$ in case (I). By Proposition 3.27, the nature of the subsequent generating pair $(\gamma_\mathrm{b}, \gamma_\mathrm{a}) = (\gamma_1^{(n+1)}, \gamma_2^{(n+1)})$ is decided by the length $l_0 = \ell(A_{\gamma_j\gamma_{j+1}^{-q}} \cap \gamma_{j+1}.A_{\gamma_j\gamma_{j+1}^{-q}})$:

- If $l_0 \geq \frac{m_j-(q+1)m_{j+1}}{2}$, Proposition 3.27 shows that $\gamma_\mathrm{b} = \gamma_j\gamma_{j+1}^{-q-1}$ is elliptic, and distance $d(A_{\gamma_\mathrm{a}}, T_{\gamma_\mathrm{b}}) = d(A_{\gamma_{j+1}}, T_{\gamma_j\gamma_{j+1}^{-q-1}}) = \frac{m_j-(q+1)m_{j+1}}{2}$. This corresponds to case (2a)(i) in Theorem 3.3.
- If $l_0 < \frac{m_j-(q+1)m_{j+1}}{2}$, Proposition 3.27 shows that $\gamma_\mathrm{b} = \gamma_j\gamma_{j+1}^{-q-1}$ is hyperbolic, with translation length $m_\mathrm{b} = m_{\gamma_j\gamma_{j+1}^{-q-1}} = m_j - (q+1)m_{j+1} - 2l_0$, and distance $d(A_{\gamma_\mathrm{a}}, A_{\gamma_\mathrm{b}}) = d(A_{\gamma_{j+1}}, A_{\gamma_j\gamma_{j+1}^{-q-1}}) = l_0$. This corresponds to case (2a)(ii) in Theorem 3.3.

Termination in case (IV). If the reduction terminates at a candidate triple in case (IV), Proposition 3.46 shows that $l \geq m_1 + m_2 - \gcd(m_1, m_2)$, and the algorithm terminates at the generating pair $(\gamma_\mathrm{a}, \gamma_\mathrm{b}) = (\gamma_1^{(N)}, \gamma_2^{(N)}) = (\gamma_k, \gamma_{k+1})$, corresponding to the candidate triple $(l^{(N)}, m_1^{(N)}, m_2^{(N)}) = (l - m_1 - m_2 + \gcd(m_1, m_2), \gcd(m_1, m_2), 0)$. Proposition 3.38 shows that γ_{k+1} is elliptic, and the length $\ell(A_{\gamma_k} \cap T_{\gamma_{k+1}}) = l^{(N)} = l - m_1 - m_2 + m_k$. Proposition 3.19 shows that v^+ remains as an endpoint of the axes intersection before the last reduction step, and Proposition 3.38 shows that it remains as an endpoint of the axis-tree intersection at the end. This corresponds to case (3) in Theorem 3.3 if the inequality is strict. When equality holds in the inequality, l satisfies the condition for case (2a) with $(j,q) = (k-1, q_{k-1}-1)$. In this situation, the quantity

$$\frac{m_j - (q+1)m_{j+1}}{2} = \frac{m_{k-1} - q_{k-1}m_k}{2} = 0, \tag{3.58}$$

thus case (2a)(i) in Theorem 3.3 applies, with the correct generators $(\gamma_k, \gamma_{k-1}\gamma_k^{-q_{k-1}}) = (\gamma_k, \gamma_{k+1})$ and correct axis-tree distance $d = 0$.

This exhaustive case analysis completes the proof of the theorem.

$\boxed{\textbf{Theorem 3.3}}$

4. Weighted Trees

In this section we extend our classification to weighted trees, where translation lengths are allowed to be arbitrary positive real numbers. We prove an analogue of Theorem 3.3 in this setting, with a dichotomy governed by the rational or irrational nature of the ratio m_2/m_1. In the irrational case, the exceptional intersection lengths forming non-free subgroups constitute a discrete subset of $(0, m_1 + m_2)$ and are shown to coincide with the gap lengths appearing in the three-gap theorem. The analysis adapts the Nielsen reduction procedure to the weighted metric setting and exploits recurrence properties of continued fractions, leading to Theorem 4.8.

We now broaden our scope from metric trees (where every edge has length 1) to the more general setting of *weighted trees*. This allows us to consider automorphism groups of a wider class of geometric objects.

Definition 4.1. Let X be a (combinatorial) tree and let

$$w : \mathcal{E}(X) \to \mathbb{R}_{>0} \tag{4.2}$$

be a function assigning a positive real number $w(e)$ to each edge e. The pair (X, w) is called a *weighted tree*.

We can metricize the tree (X, w) by regarding the weight $w(e)$ as the *length* of each edge $e \in \mathcal{E}(X)$:

Definition 4.3. The *length* of a geodesic path P, denoted by $\ell(P)$, is the sum of the weights of its constituent edges. This induces a natural metric on the vertex set, $d(v, w) = \ell([v, w])$.

It is useful to consider the full metric realization of the weighted tree:

Definition 4.4. The *geometric realization* $\mathrm{Real}(X, w)$ of a (symmetric directed) weighted tree is the metric graph obtained by identifying each combinatorial edge $e \in \mathcal{E}(X)$ with a closed interval of length $w(e)$ and gluing at vertices in the obvious way:

$$\mathrm{Real}(X, w) = \left(\mathcal{V}(X) \sqcup \left(\bigsqcup_{e \in \mathcal{E}(X)} e \times [0, w(e)] \right) \right) / \sim, \tag{4.5}$$

where the equivalence relation $\sim$ identifies, for each edge $e \in \mathcal{E}(X)$, the point $e \times \{0\}$ with α_e and the point $e \times \{w(e)\}$ with ω_e.

Definition 4.6. A *(metric) automorphism* of the weighted tree (X, w) is an isometry

$$g : \mathrm{Real}(X, w) \to \mathrm{Real}(X, w) \tag{4.7}$$

of the metric space $\mathrm{Real}(X, w)$. The group formed by the automorphisms is denoted by $\mathrm{Aut}(X, w)$.

4.1. **Classification Theorem for Weighted Trees.** Similarly to the unweighted case, we can ask when two hyperbolic automorphisms γ_1, γ_2 of a weighted tree (X, w) generate a free group. Their translation lengths $m_1 = m_{\gamma_1}$, $m_2 = m_{\gamma_2}$ and the length of the intersection of their axes $l = \ell(A_{\gamma_1} \cap A_{\gamma_2})$ are now positive real numbers. We adopt the same normalizations and definitions for the sequences $\{m_i\}$, $\{q_i\}$, and $\{\gamma_i\}$ as in the unweighted case, defined by the continued fraction expansion of $\alpha = m_2/m_1$.

We derive the following result, which generalizes Theorem 3.3:

Theorem 4.8. *Let γ_1, γ_2 be hyperbolic automorphisms of a weighted tree (X, w) with geometric triple (l, m_1, m_2) and let $\alpha = m_2/m_1$.*

If α is rational, then the sequence $\{m_i\}$ is finite and the group $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ satisfies precisely the conclusion of Theorem 3.3.

If α is irrational, then the sequence $\{m_i\}$ is infinite and Γ falls into one of the following cases:

(1) If $l = 0$, then Γ is a free group of rank two.

(2) If $0 < l < m_1 + m_2$, there are unique integers j, q with

$$(m_1 + m_2) - m_j + (q-1)m_{j+1} < l \leq (m_1 + m_2) - m_j + q m_{j+1}, \ j \geq 1, \ 0 \leq q \leq q_j - 1. \tag{4.9}$$

(a) If $l = (m_1 + m_2) - m_j + q m_{j+1}$, then $\gamma_j \gamma_{j+1}^{-q}$ is hyperbolic; set

$$l_0 := \ell(A_{\gamma_j \gamma_{j+1}^{-q}} \cap \gamma_{j+1}.A_{\gamma_j \gamma_{j+1}^{-q}}). \tag{4.10}$$

- *If $l_0 \geq \frac{m_j - (q+1)m_{j+1}}{2}$, then Γ is not free.*
- *If $l_0 < \frac{m_j - (q+1)m_{j+1}}{2}$, then Γ is free of rank two.*

(b) If $l < (m_1 + m_2) - m_j + q m_{j+1}$, then Γ is free of rank two.

(3) It is impossible to have $l \geq m_1 + m_2$.

Except for the impossible case (3) for irrational α, the generating pair $(\gamma_{\mathrm{a}}, \gamma_{\mathrm{b}})$, their translation lengths, and the relevant intersection-length or minimum-distance data match those in Table 3.7.

The proof follows the same conceptual structure as that of Theorem 3.3. We perform the same reduction algorithm for the generating pair (γ_1, γ_2) as well as the associated geometric triple (l, m_1, m_2). The same outcome as in Lemma 3.43 and Proposition 3.46 is expected if m_1 and m_2 are commensurable, and a key difference occurs if they are not.

Lemma 4.11. *If $m_2/m_1 \in \mathbb{Q}$, the change of the translation lengths agrees with the description in Lemma 3.43, where $m_k = \gcd(m_1, m_2)$ is the greatest common divisor for commensurable real numbers. The algorithm terminates in either case (II), (III), or (IV) for l under the same conditions as in Proposition 3.46.*

If $m_2/m_1 \notin \mathbb{Q}$, the change of the translation lengths agrees with a Euclidean algorithm that does not terminate. Nevertheless, for $m_2 < l < m_1 + m_2$, there are unique integers j, q with

$$(m_1+m_2) - m_j + (q-1)m_{j+1} < l \le (m_1+m_2) - m_j + qm_{j+1},\ j \ge 1,\ 0 \le q \le q_j - 1. \tag{4.12}$$

In this scenario, the algorithm terminates in either case (II) or case (III), depending on whether the inequality is strict.

Proof. The only nontrivial thing to show is the guaranteed termination for $m_2/m_1 \notin \mathbb{Q}$ and $m_2 < l < m_1 + m_2$. By the definition of the sequence $\{m_j\}$ and the irrationality of the ratio, we have that $\{m_j + m_{j+1}\}$ is strictly decreasing, and

$$\lim_{j\to\infty} (m_j + m_{j+1}) = 0. \tag{4.13}$$

Hence, for any $m_2 < l < m_1 + m_2$, there exists an index $j \ge 1$, satisfying

$$m_j - (q_j - 1)m_{j+1} = m_{j+1} + m_{j+2} \le m_1 + m_2 - l < m_j + m_{j+1}. \tag{4.14}$$

Consequently, there further exists an integer $0 \le q \le q_j - 1$, such that l satisfies the claimed inequality. The termination in either case (II) or case (III) follows straightforwardly from this fact.

□

It remains to show that the condition $m_2/m_1 \notin \mathbb{Q}$ with $l \ge m_1 + m_2$ is impossible as a geometric triple (l, m_1, m_2) for a generating pair in $\mathrm{Aut}(X, w)$.

Lemma 4.15. *Suppose (X, w) is a* **locally finite** *weighted tree, (γ_1, γ_2) is a pair of hyperbolic automorphisms in $\mathrm{Aut}(X, w)$, (l, m_1, m_2) the associated geometric triple, and $m_2/m_1 \notin \mathbb{Q}$. Then one must have $l < m_1 + m_2$.*

Proof. Assume the opposite, that $l \ge m_1 + m_2$. Let $u = \gamma_1\gamma_2.v$, then $d(v^-, u) = m_1 + m_2 \le l$, and $[v^-, u] \subset A_{\gamma_1} \cap A_{\gamma_2}$. We will derive a contradiction by constructing an infinite sequence of distinct vertices $\{v_i\}_{i=0}^{\infty}$ of valence ≥ 3 on the finite geodesic segment $[v^-, u]$:

$$\begin{aligned} & v_0 = v^-. \\ \text{For odd } i: \quad & v_i = \gamma_1^q \cdot v_{i-1}, \quad \text{where } q = \left\lfloor \frac{d(v_{i-1}, u)}{m_1} \right\rfloor. \\ \text{For even } i: \quad & v_i = \gamma_2^{-q} \cdot v_{i-1}, \quad \text{where } q = \left\lfloor \frac{d(v_{i-1}, v^-)}{m_2} \right\rfloor. \end{aligned}$$

The length $\ell([v^-, u]) = m_1 + m_2$ ensures that $d(v_{i-1}, u) \ge m_1$ when i is odd and $d(v_{i-1}, v^-) \ge m_2$ when i is even. Thus, the integer q is always at least 1, and each step moves the point a positive distance along the geodesic.

By induction on i, one sees that $d(v^-, v_i) = m_1(k_1) + m_2(k_2)$ for some integers $k_1, k_2 \geq 0$ depending on i. Since $\frac{m_2}{m_1}$ is irrational, these distances are all distinct (otherwise we get $m_1(k_1 - k_1') = m_2(k_2' - k_2)$). Hence all the points v_i are distinct.

However, this constructs an infinite set of distinct vertices with valence ≥ 3 in the compact interval $[v^-, u]$, which is impossible for a combinatorial tree. This contradiction forces us to reject the initial assumption, proving that $l < m_1 + m_2$ must hold.

□

Proof of Theorem 4.8. Lemma 4.15 implies that the condition $m_2/m_1 \notin \mathbb{Q}$ and $l \geq m_1 + m_2$ does not occur. This corresponds to irrational case (3) in Theorem 4.8.

Except for the case excluded above, we perform the same algorithm for (γ_1, γ_2) and the associated geometric triple (l, m_1, m_2) as described in Section 3. By Lemma 4.11, the algorithm terminates in case (II), (III), or (IV) within finitely many reduction steps. Propositions 3.12, 3.19, 3.27, and 3.38 then imply our claim, similarly to the proof of Theorem 3.3:

- When $l = 0$, the algorithm terminates in case (II) instantly, which corresponds to case (1) in Theorem 4.8.
- When $0 < l < m_2$, the algorithm terminates in case (III) instantly, which corresponds to case (2b) in Theorem 4.8 with $j = 1$ and $q = 0$.
- When $m_2 \leq l < m_1 + m_2 - \gcd(m_1, m_2)$ (if m_1 and m_2 are commensurable) or $m_2 \leq l < m_1 + m_2$ (if incommensurable), the algorithm terminates in case (II) or case (III) after reduction, which corresponds to case (2a)(i), (2a)(ii) or (2b) for certain integers j and q.
- When m_1 and m_2 are commensurable and $l \geq m_1 + m_2 - \gcd(m_1, m_2)$, the algorithm terminates in case (IV), which corresponds to case (3), or case (2a)(i) with $(j, q) = (k - 1, q_{k-1} - 1)$.

Theorem 4.8

As an interesting remark, the exceptional lengths in Theorem 4.8 that determine the group's structure are intimately related to the *three-gap theorem* (also known as the Steinhaus conjecture)[11]. This theorem states that for any irrational number α and positive integer N, the fractional parts $\{i\alpha - \lfloor i\alpha \rfloor \mid i = 1, \ldots, N\}$ partition the circle $\mathbb{S}^1 = [0,1]/\sim$ into intervals with at most three distinct lengths. As N varies, the set of all gap lengths that appear for a fixed α is countable:

Proposition 4.16 ([5], Theorem 2)**.** *Let $0 < \alpha < 1$ be an irrational number with continued fraction expansion*

$$\alpha = [0; q_1, q_2, q_3, \ldots]. \tag{4.17}$$

Let $\alpha_1 = \alpha$ and define

$$\alpha_{i+1} = \alpha_i^{-1} - \lfloor \alpha_i^{-1} \rfloor, \ \forall i \in \mathbb{N}_+. \tag{4.18}$$

Then, the set of distinct gap lengths arising in the three-gap theorem for α consists of values

$$(1 - q\alpha_j) \prod_{i=1}^{j-1} \alpha_i, \tag{4.19}$$

for any $j \geq 1$ and $0 \leq q \leq q_j - 1$.

A connection between Theorem 4.8 and the three-gap theorem arises from the Euclidean algorithm and continued fractions:

Corollary 4.20. *In the context of Theorem 4.8, let $\alpha = m_2/m_1$ be irrational, and let*

$$\mathcal{L} = \{(m_1 + m_2) - m_j + qm_{j+1} \,|\, j \geq 1,\ 0 \leq q \leq q_j - 1\} \tag{4.21}$$

be the set of exceptional intersection lengths*. Then the normalized set*

$$\{1 + \alpha - (l/m_1) \,|\, l \in \mathcal{L}\} \tag{4.22}$$

is exactly the set of all gap lengths that occur in the three-gap theorem for the irrational number α.

Proof. Define $\alpha_j = m_{j+1}/m_j$ for $j \geq 1$, so that $\alpha_1 = \alpha$. The recurrence relation for the Euclidean algorithm becomes:

$$q_j = \lfloor \alpha_j^{-1} \rfloor, \quad \alpha_{j+1} = \alpha_j^{-1} - q_j. \tag{4.23}$$

This is precisely the recurrence defining the continued fraction expansion of α:

$$\alpha = [0; q_1, q_2, q_3, \dots]. \tag{4.24}$$

Now, consider an element of $\mathcal{L}$:

$$l_{j,q} = (m_1 + m_2) - m_j + qm_{j+1}. \tag{4.25}$$

It corresponds to the normalized value

$$1 + \alpha - \frac{l_{j,q}}{m_1} = \frac{(m_1 + m_2) - l_{j,q}}{m_1} = \frac{m_j - qm_{j+1}}{m_1}. \tag{4.26}$$

We can express this quantity in terms of the α_i:

$$\frac{m_j - qm_{j+1}}{m_1} = (1 - q\alpha_j) \cdot \frac{m_j}{m_1} = (1 - q\alpha_j) \prod_{i=1}^{j-1} \alpha_i, \tag{4.27}$$

for $j \geq 1$ and $0 \leq q \leq q_j - 1$. As in Proposition 4.16, these values represent the distinct gap lengths arising in the three-gap theorem for the irrational number $\alpha = [0; q_1, q_2, \dots]$. This completes the identification.

□

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