Division of
Academic Enhancement
UNIVERSITY OF GEORGIA

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MATH 1113 Exam 2 Review

Topics Covered

Section 3.1: Inverse Functions

Section 3.2: Exponential Functions

Section 3.3: Logarithmic Functions

Section 3.4: Properties of Logarithms

Section 3.5: Exponential and Logarithmic Equations and Applications

Section 3.6: Modeling with Exponential and Logarithmic Functions

How to get the most out of this review:

1. **Watch the video and fill in the packet for the selected section. (Video links can be found at the two web addresses at the top of this page)**
2. **After each section there are some 'Practice on your own' problems. Try and complete them immediately after watching the video.**
3. **Check your answers with the key on the last page of the packet.**
4. **Go to office hours or an on-campus tutoring center to clear up any 'muddy points'.**

Section 3.1: Inverse Functions

Functions

Recall: A function from A to B is a rule of correspondence that assigns each element of A to exactly ONE element of B.

A one to one function from A to B is a rule of correspondence that assigns each element of A to exactly ONE element of B ***AND*** each element of B is assigned to exactly ONE element of A. All one to one functions are invertible.

Vertical line test

A graph in the xy-plane represents a function $y = f(x)$ provided that any vertical line intersects the graph in at most one point.

Horizontal line test

A graph of a *function* in the xy plane represents a one to one function $y = f(x)$ provided that any horizontal line intersects the graph in at most one point.

Inverse Function

A function $y = f^{-1}(x)$ is an inverse function of $f(x)$ if the following properties hold true.

- The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected over the line $y = x$.
- The compositions of $f(x)$ and $f^{-1}(x)$ returns x. $f^{-1}(f(x)) = x = f(f^{-1}(x))$
- The x and y coordinates of any point on $f(x)$ are reversed as points on $f^{-1}(x)$. For example, if (4,3) is a point on $f(x)$, then (3,4) is a point on $f^{-1}(x)$.
- The domain of $f(x)$ is the range of $f^{-1}(x)$ and the range of $f(x)$ is the domain of $f^{-1}(x)$
- All one to one functions have inverses.
- Functions that are not one to one functions *may* have inverses if the domain of $f(x)$ can be restricted to make it one to one. (Example $y = x^2$)

Finding the inverse function algebraically

- Write the original equation of $f(x)$ as $y = f(x)$
- Rewrite the equation replacing y with x and vice versa
- Solve for y
- If the new equation is a function, then it is the inverse of $f(x)$ (Note a domain restriction may be required)

Example 1

Let $f(x)$ be an invertible function. Complete the following table

x	-2	-1	0	1	4	5	6
$f(x)$	10	7	4	0	-3	-4	-7

x							
$f^{-1}(x)$							

Example 2

Use the function $f(x)$ to answer the following:

$$f(x) = \frac{4}{x-2}$$

(a) Find the inverse of $f(x)$. (Find $f^{-1}(x)$)

(b) Determine the domain and range of $f(x)$.

(c) Is $f(x)$ a one to one function? Why do you think so?

Example 3

Answer the following about inverses.

(a) Is $f(x) = x^2$ invertible? If so, where and find its inverse.

(b) What about $f(x) = (x - 2)^2 + 5$?

(c) What can we say about the invertibility of quadratic functions?

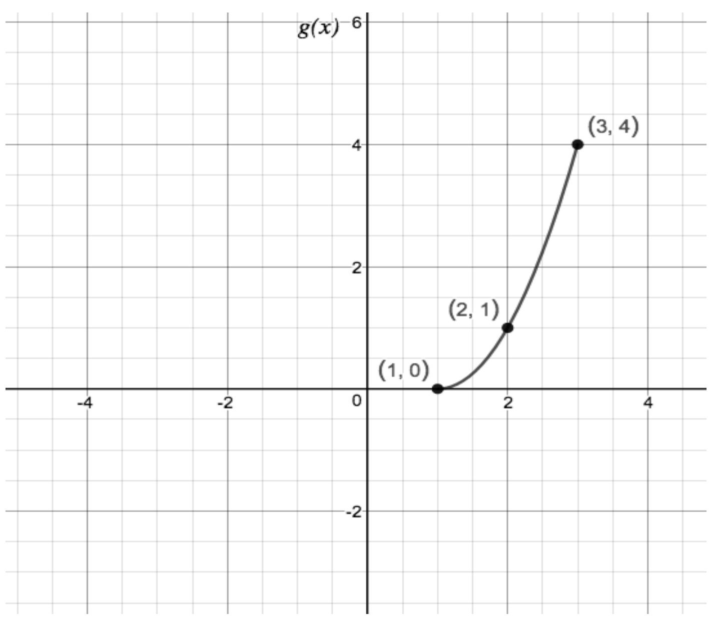
Practice on Your Own

1. Find the inverse, domain and range of

   (a) $f(x) = \frac{2x+5}{7-3x}$

   (b) $g(x) = \frac{x+1}{x-2}$

2. The graph of $g(x)$ is given below. Sketch its inverse. You must plot and write the corresponding points on the inverse function.



Section 3.2: Exponential Functions

Properties of Exponents

$b^0 = 1$ $\quad b^n b^m = b^{n+m}$ $\quad \frac{b^n}{b^m} = b^{n-m}$ $\quad (b^n)^m = b^{nm} = (b^m)^n$

$(ab)^n = a^n b^n$ $\quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $\quad b^{-n} = \frac{1}{b^n}$ $\quad \frac{1}{b^{-n}} = b^n$

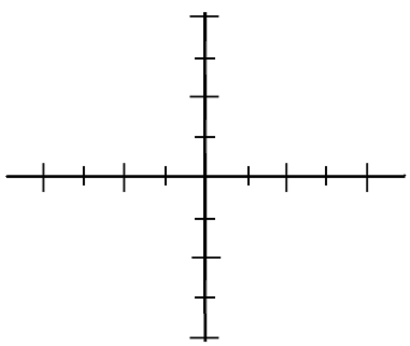
Exponential Functions vs. Power Functions

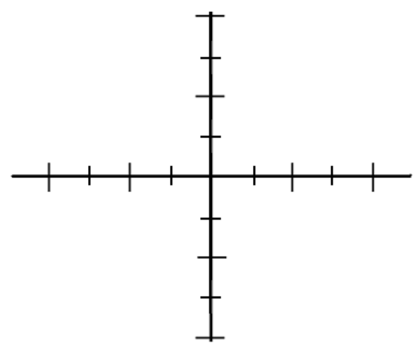
A power function is a variable raised to a constant power like $y = x^2$ (variable is the base)

An exponential function is a constant base raised to a variable power like $y = 2^x$
(variable is in the exponent)

The general form of an exponential function is $y = b^x$ where b is the base of the function and $b \neq 1$.

- If $0 < b < 1$, then $y = b^x$ will decrease exponentially
- If $1 < b$, then $y = b^x$ will increase exponentially





Example 4
Solve for x.

$$27^{3x+2} = 81^{4x+5}$$

Example 5

Determine the domain of the function $f(x) = \sqrt{1 - e^{3x-9}}$

Practice on Your Own

1. Solve the following for x.

$$2^{x+1} = 8^x$$

2. Combine the following expression into a single exponential.

$$\frac{13^{x-1}(13^y)^2}{13^2}$$

3. For the following function, determine the domain, range, any asymptotes, x and/or y intercepts and if the function is increasing or decreasing.

$$g(x) = 3^{-x} - 1$$

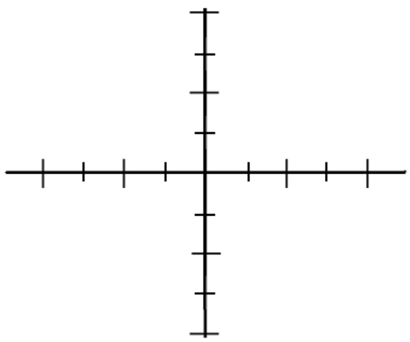
4. Let $f(x) = 7a^x + c$, where a is positive. If $f(x)$ has a horizontal asymptote at $y = 0.68$ and contains the point $(2,4)$, determine the values for a and c.

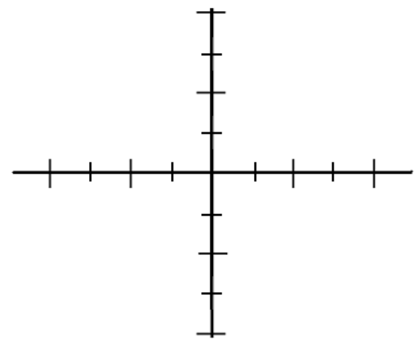
Section 3.3: Logarithmic Functions

Logarithmic Functions
A logarithmic function is the inverse function of an exponential function.

$$y = b^x \rightarrow \log_b y = x$$

$y = b^x$	$\log_b y = x$
$D: (-\infty, \infty)$	$D: (0, \infty)$
$R: (0, \infty)$	$R: (-\infty, \infty)$





Example 6
Find all the solutions to the equation $e^{4x-1} = 6$.

Example 7
Find all the solutions to the equation $e^{2x} - 10e^{x} + 21 = 0$.

Example 8
Determine the domain of the function $f(x) = \log(x^2 - 9)$

Example 9
Solve for x.

$$2^x = 5^{4x+3}$$

Example 10
Solve for x.

$$2 \cdot 5^x = 2^x$$

Practice on Your Own

1. Solve for x.

$$2^{5x}5^{x+1} = e^x$$

2. For the following function, determine the domain, range, any asymptotes, x and/or y intercepts and if the function is increasing or decreasing.

$$f(x) = \ln(2x + 1)$$

Section 3.4: Properties of Logarithms

Properties of Log Functions

$$\log_b 1 = 0 \qquad \log_b b = 1 \qquad b^{\log_b x} = x$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n \qquad \log_b(mn) = \log_b m + \log_b n \qquad \log_b(m^r) = r\log_b m$$

Remember, $\ln x = \log_e x$ and $\log x = \log_{10} x$. So, the same properties hold for $\ln x$ and $\log x$.

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b} \; for\ all\ a > 0$$

Example 11
Solve for x.

$$\log(x + 21) + \log(x) = 2$$

Example 12
Solve for x.

$$\ln(3x + 7) = 3 + \ln(x)$$

Example 13

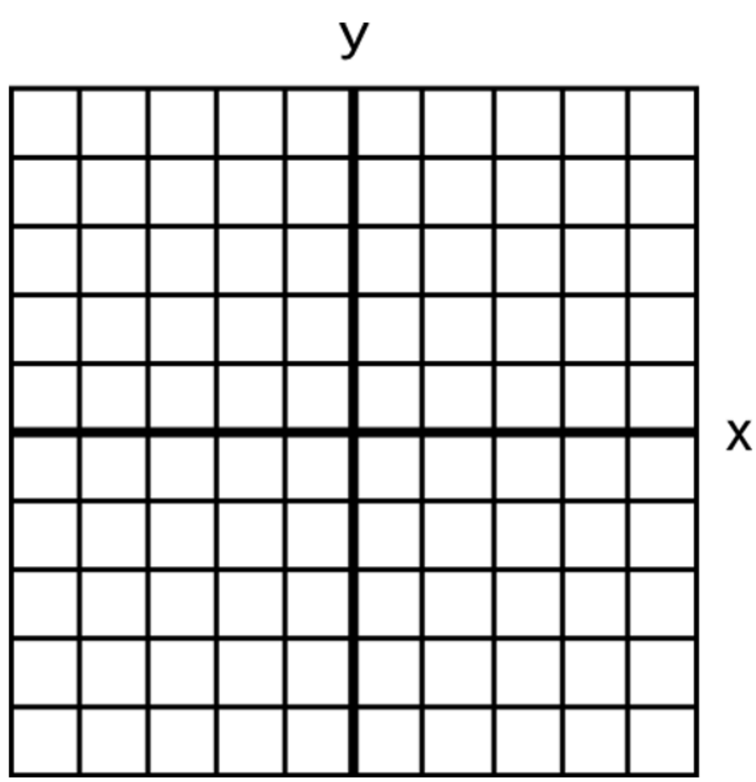
Find all the solutions of the equation $ln(x^2) = 5$.

Example 14

Find all the solutions of the equation $\log(x) + \log(x + 1) = \log 6$

Example 15

Sketch the graph of $f(x) = \log_2(x + 1) - 2$ below. Mark two points explicitly as well as its asymptote.



Practice on Your Own

1. Solve for x.

$$2\log_3(x-2) - \log_3(4x) = 2$$

2. Combine the following expression into a single logarithm.

$$2\ln(x+1) - \frac{1}{2}\ln(x-2)$$

Section 3.5: Exponential and Logarithmic Equations and Applications

Compound Interest

Compound Interest Formula (Interest Compounded n times per Year)

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$A = Amount\ of\ money\ at\ time\ t$
$P = intial\ amount\ invested\ (principal)$
$r = annual\ interest\ rate$
$t = time\ in\ years$
$n = number\ of\ times\ per\ year$

Compound Interest Formula (Interest Compounded Continuously)

$$A = Pe^{rt}$$

$A = Amount\ of\ money\ at\ time\ t$
$P = intial\ amount\ invested\ (principal)$
$r = annual\ interest\ rate$
$t = time\ in\ years$

Example 16

You are going to invest money in a savings account that compounds interest quarterly. If you want the money to double in value after 12 years, what is the minimum interest rate you must find? (Round your answer to 4 decimal places.)

Practice on Your Own

1. Money is invested at interest rate r, compounded continuously. Express the time required for the money to quadruple, as a function of r.

2. A savings fund pays interest at rate of 5% per year. How much money should be invested not to yield $1000 in the fund after 10 years if the interest is compounded
   (a) monthly? (Round your answer to the nearest cent)

   (b) continuously? (Round your answer to the nearest cent)

3. Two investments are started at the same time. The first investment begins with $5200 and earns interest compounded continuously at the rate of 7.6% per annum. The second account begins with $6400 and earns interest compounded continuously at a rate of 4.3% per annum.
   (a) Determine the time T (in years) for the values of the two accounts to be the same.

   (b) Determine the common value of the accounts at that time. (Round your answer to the nearest cent)

<u>Section 3.6: Modeling with Exponential and Logarithmic Functions</u>

When building an exponential model pay close attention to your initial value (if provided), your rates (is it decreasing or increasing?), units of time (as applicable) and your base (for continuous growth use base e).

Write the appropriate equation and then fill in any known values. Use any data points provided to solve for the remaining constants as needed.

Example 17
Jennifer is terrified of spiders. Much to her horror, she finds a spider nest in her basement and lays down some poison traps. The population of spiders is supposed to decrease exponentially if the poison works properly. If it takes 12 hours for the first 10% of spiders to die, how long until 75% of the spiders are gone? (Round your answer to the nearest hour)

Example 18

You have two job options. ACME offers you a base pay of \$35,000 per year, plus \$10,000 in special company stock whose value grows at a continuously compounded rate of 8%. Turtle Wax, Inc. offers you \$40,000 per year, but no stock.

(a) Let T be the cumulative value of your compensation t years after you start working for Turtle Wax, Inc. To be clear, the value T should be cumulative, so for example when $t = 1$ you should get $T = 40{,}000$ and when $t = 2$, you should get $T = 80{,}000$. Find an equation which expresses T as a function of t.

(b) Let A be the cumulative value of your total compensation (salary paid + stock value) at ACME t years after you were hired. In other words, A is the total cash you've earned in salary + the value of your stock at time t. Find an equation which expresses A as a function of t.

(c) Assuming you earn no raises from either company, which is the better to work for if you only plan to stay for one year? What about two years? What about 10 years?

Example 19

A certain radioactive isotope is present in organic materials and is commonly used for dating historical artifacts. The isotope has a decay rate of 0.000121. What is its half-life? (Round answer to the nearest 10)

Example 20

For each of the following "real life" situations, state if it is best modeled by an exponential, logarithmic, linear, or quadratic function. Don't say "none" and then come up with philosophical excuses. Just tell me one.

(a) You come to a casino with $500 and play black jack. You bet $10 on every hand and lose every time until the money is gone. What kind of function best describes your money as a function of time?

(b) You come to a casino with $500 and play black jack. You bet half of the money you have left on every hand and lose every time until the money is gone. What kind of function best describes your money as a function of time?

(c) You want to build a circular enclosure and fencing costs $2 per linear foot. What kind of function best describes the area of the enclosure as a function of how much money you spend on fencing?

(d) You are staging an athletic tournament. Every round you pair off all the teams and the losing teams are eliminated. Let x be the number of teams in your tournament and let y be the number of rounds that need to be played before the championship game. What kind of function best describes y as a function of x?

(e) Your faucet has been left on and is pouring water into your parabolic shaped sink. What kind of function best describes the total volume of water in your sink as a function of time?

Practice on Your Own

1. A certain isotope has a half-life of 57 years. If 40g of the isotope are initially present, how long will it take for only 30g of the isotope to remain? (Round your answer to 4 decimal places.)

2. You've got a fever, a fever for logarithms. Even though it's incurable, the doc tried to give you some medicine to treat it. You took 300 milligrams of this totally ineffective medicine three hours ago, and only 100 milligrams of it are left in your body now. Assuming the decay in this medicine is exponential, how much longer will it be before there is only 10 milligrams of the medicine left?

3. A culture of bacteria initially has 400 bacteria present. 10 hours later the bacteria population has grown to 1275.
   (a) How many bacteria were present after 8 hours?

   (b) When will the population reach 3000 bacteria?

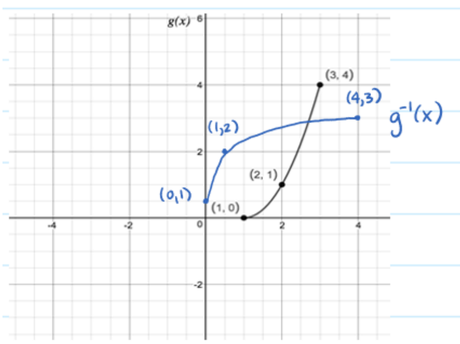
Answers to Practice Problems

Special comment about the format of answer key: Some answers were approximated with decimals for reference only! There are many different forms of exact correct answers depending on how far you choose to simplify or which correct algebra path you take. NEVER approximate a final answer on exams unless the problem explicitly instructs you to round to a specific decimal place.

Section 3.1

1. Inverse, domain and range
   (a) $f^{-1}(x) = \frac{7x-5}{3x+2}$, $D: \left(-\infty, \frac{7}{3}\right) \cup \left(\frac{7}{3}, \infty\right)$, $R: \left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$

   (b) $g^{-1}(x) = \frac{2x+1}{x-1}$, $D: (-\infty, 2) \cup (2, \infty)$, $R: (-\infty, 1) \cup (1, \infty)$
2.



Section 3.2

1. $x = \frac{1}{2}$
2. 13^{x+2y-3}
3. $D: (-\infty, \infty)$, $R: (-1, \infty)$, Asymptote: $y = -1$, x/y intercept: $(0,0)$, Decreasing
4. $f(x) = 7\left(\sqrt{\frac{3.32}{7}}\right)^x + 0.68$

Section 3.3

1. Using log base e, (ln):

$$x = \frac{-\ln 5}{5\ln 2 + \ln 5 - 1} = \frac{\ln 5}{1 - 5\ln 2 - \ln 5}$$

Using log base 2, $\log_2$:

$$x = \frac{-\log_2 5}{5 + \log_2 5 - \log_2 e} = \frac{\log_2 5}{\log_2 e - \log_2 5 - 5}$$

Using log base 5, $\log_5$:

$$x = \frac{-1}{1 + 5\log_5 2 - \log_5 e} = \frac{1}{\log_5 e - 5\log_5 2 - 1}$$

2. $D: \left(-\frac{1}{2}, \infty\right)$, $R: (-\infty, \infty)$, Asymptote: $x = -\frac{1}{2}$, x/y intercept: $(0,0)$, Increasing

Section 3.4

1. $x = \frac{40+\sqrt{1584}}{2} \approx 39.9$ <u>Note</u>: The negative case using the quadratic equation is an invalid solution.

2. $\ln \frac{(x+1)^2}{\sqrt{x-2}}$

Section 3.5

1. $t = \frac{\ln(4)}{r}$

2. Savings fund
   (a) $607.16
   (b) $606.53

3. Two investments
   (a) $t = \frac{1}{0.033} \ln\left(\frac{6400}{5200}\right) \approx 6.29\ years$
   (b) $8388.47

Section 3.6

1. $t = \frac{57 \ln(3/4)}{\ln(1/2)} \approx 23.6571\ years$

2. $t = \frac{3 \ln(1/30)}{\ln(1/3)} - 3 \approx 6.29\ hours\ later$

Note: You are solving for the additional time after you measured 100mg 3 hours after the initial dose of 300mg.

3. Bacteria

   (a) $P = 400e^{\frac{8}{10}\ln\left(\frac{1275}{400}\right)} = 400\left(\frac{1275}{400}\right)^{\frac{8}{10}} \approx 1011.16$

   (b) $t = \frac{10 \ln(30/4)}{\ln(1275/400)} \approx 17.38\ hours$