UNIVERSITY OF GEORGIA

DEPARTMENT OF MATHEMATICS
MATH 2250 - PRACTICE FINAL
FALL 2023

PRINTED NAME : Solutions

STUDENT ID :

DATE : ___/___/___

GRADE
/245

INSTRUCTOR :

SECTION :

Nº	SCORE	MAX
1		30
2		5
3		5
4		5
5		5
6		5
7		5
8		5
9		5
10		5
11		5
12		5
13		5
14		5
15		5
16		5
17		10
18		10
19		10
20		10
21		10
22		10
23		10
24		10
25		10
26		10
27		10
28		10
29		10
30		10
TOTAL		245

INSTRUCTIONS

- This is a set of practice problems that can be used to review for the final exam. The real exam follows the same format, but is shorter.
- The exam has two parts: the first one consists of Multiple Choice (MC) questions, and the second part of Free Response (FR) ones. You must show work for both parts. An unjustified answer will receive no credit. If you are using a shortcut, explain it.
- Please show your work in a neat and organized manner. Circle the answer for MC questions and put a box around the final answer for the FR questions.
- Smart devices (including smart watches and cell phones) are not allowed and may not be on your person.
- You do not need a calculator; answers containing symbolic expressions such as $\cos(\pi/3)$ and $\ln(e^4)$ are acceptable. Include an exact answer for each problem.
- If you plan to use a calculator, you are only allowed to use a TI-30XS Multiview calculator; the name must match exactly. No other calculators or sharing of calculators is allowed.

Part I: Multiple Choice

Show your work and **circle** your answer.

1. [30 pts] Find the following limits. If they do not exist, choose DNE.

form $\left[\frac{0}{2}\right]$

_____ PTS (a) $\lim_{x\to 0} \dfrac{\sin 3x}{\cos(x)+\cos(2x)}$ = 0

(A) DNE (C) 3 (E) -3

(B) 0 (D) -1

______PTS (b) $\lim_{x\to 2^-} \dfrac{x^2-5x+6}{x-2}$

Ⓐ DNE Ⓒ -1 Ⓔ 4

Ⓑ 0 Ⓓ 1

$\left[\frac{0}{0}\right]$

$= \lim_{x\to 2^-} \dfrac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}}$

$= \lim_{x\to 2^-} (x-3)$

$= -1$ //

______PTS (c) $\lim_{x\to 4} \frac{\sin(x-4)}{\tan(x^2-16)}$

(A) DNE
(B) 0
(C) $\frac{1}{8}$
(D) $\frac{1}{16}$
(E) $\frac{1}{32}$

$\left[\frac{0}{0}\right]$

$$= \lim_{x\to 4} \frac{\sin(x-4)}{(x-4)} \cdot \frac{(x^2-16)}{\sin(x^2-16)} \cdot \frac{\cos(x^2-16)\,(x-4)}{(x^2-16)}$$

$\to 1 \qquad \to 1$

$$= \lim_{x\to 4} \frac{\cos(x^2-16)\,\cancel{(x-4)}}{\cancel{(x-4)}\,(x+4)}$$

$\to 1 \qquad \to 8$

$$= \frac{1}{8} \;/\!/$$

Or faster if using L'Hôpital:

$\left[\frac{0}{0}\right]$ L'Hôpital

$$= \lim_{x\to 4} \frac{\overbrace{\cos(x-4)}^{\to 1}}{\underbrace{2x}_{\to 8}\;\underbrace{\sec^2(x^2-16)}_{\to 1}}$$

$$= 1/8 \;/\!/$$

________ PTS (d) $\lim_{h\to 0} \dfrac{\sin(\pi+h)-\sin(\pi)}{h}$

- (A) DNE
- (B) 1
- (C) $\frac{-\sqrt{2}}{2}$
- (D) 0
- (E) -1

$$= (\sin x)'\Big|_{x=\pi}$$

$$= \cos x\Big|_{x=\pi}$$

$$= -1$$

________ PTS (e) $\lim_{x\to 0^+} (x+\pi/x)^{\pi x}$

(A) $-\infty$

(B) 0

(C) 1

(D) ∞

(E) DNE

[∞^0] indeterminate

$$= e^{\lim_{x\to 0^+} \overbrace{\pi x}^{\to 0} \cdot \ln \overbrace{\left(x+\frac{\pi}{x}\right)}^{\to \infty}}$$

[$0\cdot\infty$] still indeterminate

$$= e^{\lim_{x\to 0^+} \frac{\overbrace{\ln\left(x+\frac{\pi}{x}\right)}^{\to\infty}}{\underbrace{\frac{1}{\pi x}}_{\to\infty}}}$$

[$\frac{\infty}{\infty}$] L'Hôpital

$$= e^{\lim_{x\to 0^+} \frac{\left(1-\pi/x^2\right)/\left(x+\frac{\pi}{x}\right)}{-1/\pi x^2}}$$

$$= e^{\lim_{x\to 0^+} \frac{\overbrace{-\pi x^2+\pi^2}^{\to \pi^2}}{\underbrace{x+\frac{\pi}{x}}_{\to\infty}}}$$

$$= e^0$$

$$= 1 \,//$$

______PTS (f) $\lim_{x\to+\infty} \sqrt{x^2-x} - x$

Ⓐ $-\infty$ Ⓓ 0 Ⓖ DNE

Ⓑ -1 Ⓔ $\frac{1}{2}$

Ⓒ -$\frac{1}{2}$ Ⓕ ∞

$[\infty - \infty]$ indeterminate

$$= \lim_{x\to\infty} \sqrt{x^2-x} - x \cdot \frac{\left(\sqrt{x^2-x} + x\right)}{\left(\sqrt{x^2-x} + x\right)}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2} - x - \cancel{x^2}}{\sqrt{x^2-x} + x}$$

$$= \lim_{x\to\infty} \frac{-\cancel{x}}{\cancel{x}\left(\underbrace{\sqrt{1 - 1/x} + 1}_{\to 2 \text{ as } x\to\infty}\right)}$$

since x is positive

$$= -\frac{1}{2} \;//$$

_____ PTS 2. [5 pts] Assume that $x = a$ is a vertical asymptote of $y = \dfrac{x^2 - 25}{(x-2)(x^2 - 9x + 20)}$. Find the sum of all possible values of a:

- (A) -5
- (B) 6
- (C) 5
- (D) 8
- (E) 10

$$y = \frac{(x-5)(x+5)}{(x-2)(x-5)(x-4)}$$

Candidates for vertical asymptotes: $x = 2$, $x = 5$, $x = 4$.

- $\displaystyle\lim_{x \to 2^-} \frac{(x-5)(x+5)}{(x-2)(x-5)(x-4)} = +\infty$ $\left[\frac{-21}{0}\right]$ $\therefore$ $x = 2$ is a vertical asymptote of y.

- $\displaystyle\lim_{x \to 5^-} \frac{\cancel{(x-5)}(x+5)}{(x-2)\cancel{(x-5)}(x-4)} = \frac{10}{3}$ $\left[\frac{10}{3}\right]$ $\therefore$ $x = 5$ is <u>not</u> a vertical asymptote of y (y has a hole at $x=5$)

- $\displaystyle\lim_{x \to 4^-} \frac{(x-5)(x+5)}{(x-2)(x-5)(x-4)} = -\infty$ $\left[\frac{-9}{0}\right]$ $\therefore$ $x = 4$ is a vertical asymptote of y.

Answer: $2 + 4 = 6$

______PTS 3. [5 pts] The graph of $y = \dfrac{x-5}{x^2-7x+a}$ has a vertical asymptote and a removable discontinuity. What is the value of a?

(A) 0

(B) 3

(C) 5

(D) 7

(E) 10

(F) There is not enough information to determine a

For the graph to have a removable discontinuity, $(x-5)$ has to be one of the factors of x^2-7x+a.

In order for the linear factors to add up to $-7x$, the other factor has to be $(x-2)$.

Since a is the product of the constant terms, $a=(-5)(-2)=10$.

____ PTS 4. [5 pts] Given the values of $f(x)$ and $f'(x)$ in the table below, and given that

$$g(x) = \frac{f(2x)}{f(x)},$$

find $g'(1)$.

x	f(x)	f'(x)
1	-1	½
2	2	3
3	-3	⅓
4	4	3⁄2

Ⓐ -28 Ⓒ -6 Ⓔ -4

Ⓑ -7 Ⓓ -5 Ⓕ $\frac{3}{2}$

Quotient Rule

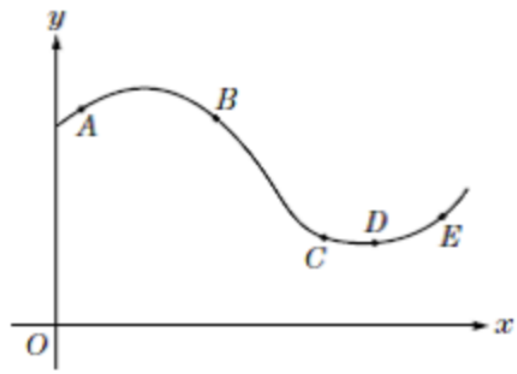
Chain Rule $(f(2x))'$

$$g'(x) = \frac{f(x)\cdot \underbrace{f'(2x)\cdot 2} - f(2x)\cdot f'(x)}{(f(x))^2}$$

$$g'(1) = \frac{\overbrace{f(1)}^{=-1}\cdot \overbrace{f'(2)}^{=3}\cdot 2 - \overbrace{f(2)}^{=2}\cdot \overbrace{f'(1)}^{1/2}}{(\underbrace{f(1)}_{=-1})^2}$$

$$= -7$$

______ PTS 5. [5 pts] At which point on the graph of $y(x)$, given below, are $y'(x)$ and $y''(x)$ both negative?



Ⓐ A   Ⓒ C   Ⓔ E

Ⓑ B   Ⓓ D

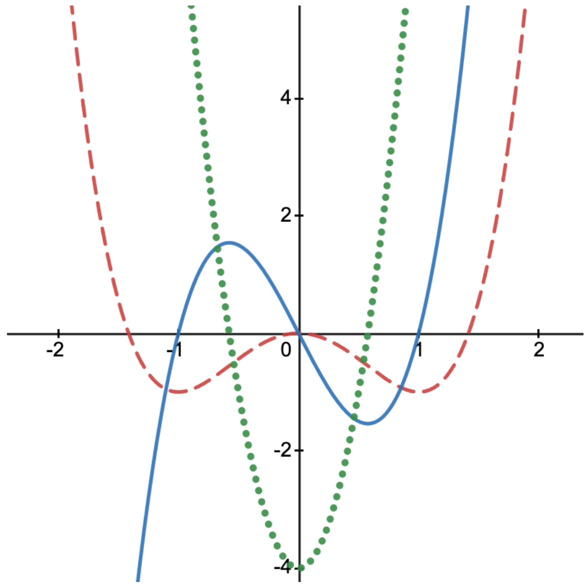
$y' < 0$ at B and C

At B, y' is decreasing $\Leftrightarrow$ $y'' < 0$

At C, y' is increasing $\Leftrightarrow$ $y'' > 0$

Thus, y' and y'' are both negative at B.

______ PTS 6. [5 pts] The functions f, f' and f'' are graphed below:



Which of the following statements is true?

(A) f is full, f' is dashed

(B) f is full, f' is dotted

(C) f is dashed, f' is full

(D) f is dashed, f' is dotted

(E) f is dotted, f' is dashed

________ PTS 7. [5 pts] (Fall 2017) Find ALL the horizontal asymptotes of the function $y = \frac{7|x|-3}{4x-12}$. If there aren't any, write DNE.

(A) DNE

(B) $y = \frac{7}{4}$ and $y = -\frac{7}{4}$

(C) $y = \frac{7}{4}$

(D) $y = -\frac{7}{4}$

(E) $y = 0$

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{7\cancel{x}}{4\cancel{x}} = \frac{7}{4}$$

$$\lim_{x \to -\infty} y = \lim_{x \to -\infty} \frac{-7\cancel{x}}{4\cancel{x}} = \frac{-7}{4}$$

$\therefore$ y has two horizontal asymptotes: $y = \frac{7}{4}$ and $y = -\frac{7}{4}$. //

______PTS 8. [5 pts] Let $f(x) = 3^{\frac{x^2-9}{x-5}}$. At which value(s) of x does this function have a local maximum?

(A) $x = 0$

(B) $x = 1$

(C) $x = 9$

(D) $x = 1$ and $x = 9$

(E) None of these

$$\text{dom } f(x) = \mathbb{R} \setminus \{5\}$$

$$f'(x) = \frac{2x(x-5) - (x^2-9)}{(x-5)^2} \cdot \ln 3 \cdot 3^{\frac{x^2-9}{x-5}}$$

$$= \frac{x^2 - 10x + 9}{(x-5)^2} \cdot \underbrace{\ln 3 \cdot 3^{\frac{x^2-9}{x-5}}}_{\text{always } > 0}$$

$= 0$: $x^2 - 10x + 9 = 0 \iff x = 1, 9$

DNE: $(x-5)^2 = 0 \iff x = 5$

x		1		5		9	
sign of f'	+	0	−	‖	−	0	+
		local maximum				local minimum	

9. [5 pts] What is the absolute maximum of the function $f(x) = 5 - 2|x-1|$ on $[-1, 2]$?

Ⓐ 1 Ⓒ 3 Ⓔ 5

Ⓑ 2 Ⓓ 4

$$f(x) = \begin{cases} -2x+7, & x \in [1,2] \\ 2x+3, & x \in [-1,1) \end{cases}$$

$$f'(x) = \begin{cases} -2, & x \in (1,2) \\ 2, & x \in (-1,1) \end{cases}$$

No critical values outside $x = 1$.

$f(1) = 5 \leftarrow$ absolute maximum

$f(-1) = 1 \leftarrow$ absolute minimum

$f(2) = 3$

$\therefore$ $f(x)$ achieves an absolute maximum value of 5 at $x = 1$.

______ PTS 10. [5 pts] Find the slope of the line tangent to the graph of

$$3x^2 + 2xy + y^2 = 2 \qquad [*]$$

at $x = 1$.

(A) -2 (C) 2 (E) not defined

(B) 0 (D) 4

- Differentiating both sides of [*] with respect to x:

$$6x + 2xy' + 2y + 2yy' = 0 \qquad [**]$$

- At $x=1$, [*] is:

$$3 + 2y + y^2 = 2$$

$$\Leftrightarrow y^2 + 2y + 1 = 0$$

$$\Leftrightarrow (y+1)^2 = 0 \Rightarrow y = -1$$

- At $(1,-1)$, [**] becomes:

$$6 + \cancel{2y'} - 2 - \cancel{2y'} = 0$$

$4 = 0$ impossible !

$\therefore$ therefore the slope of the tangent line to [*] is undefined.

______PTS 11. [5 pts] What is the absolute maximum of the function

$$f(x) = \begin{cases} -x & \text{if } x \in [-1, 0], \\ x - x^2/4 & \text{if } x \in (0, 3]? \end{cases}$$

(A) 0 (C) $\frac{3}{2}$ (E) $\frac{5}{2}$

(B) 1 (D) 2

$$f'(x) = \begin{cases} -1, & x \in [-1, 0) \\ 1 - \frac{x}{2}, & x \in (0, 3] \end{cases}$$

critical values : $x = 0$ and $x = 2$

$f(-1) = 1$
$f(0) = 0$
$f(2) = 1$
$f(3) = 3/4$

absolute maxima

Therefore, $f(x)$ achieves an absolute maximum of 1 at $x = -1$ and $x = 2$.

_____ PTS 12. [5 pts] Compute $\int_0^8 3x^2 - 4x + \frac{x^{\frac{1}{3}}}{6}\,dx$.

(A) 324

(B) 386

(C) 400

(D) 450

(E) 898

$$= \cancel{3}\frac{x^3}{\cancel{3}} - \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} + \frac{1}{\cancel{6}_{2}} \cdot \frac{\cancel{3}}{4} x^{4/3} \Big|_0^8$$

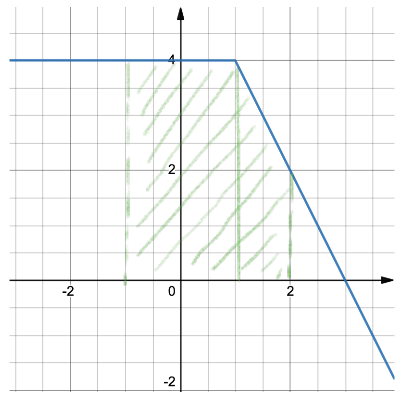
$$= \left[8^3 - 2 \cdot 8^2 + \frac{1}{8}\sqrt[3]{8^4}\right] - \left[0\right]$$

$$= \left[\underbrace{8 \cdot 8^2 - 2 \cdot 8^2}\, + \frac{1}{\cancel{8}} \cdot \cancel{8}\sqrt[3]{8}\right]$$

$$= 6 \cdot 8^2 + 2$$

$$= 386 \,//$$

______PTS 13. [5 pts] Compute $\int_{-1}^{2} 2f(x) + 1\, dx$ where the graph of f is given below.



$= 2\left(\text{Area of the shaded region}\right) + \int_{-1}^{2} 1\, dx$

$= 2\left(\text{Area of rectangle} + \text{Area of trapezoid}\right) + (2-(-1))$

$= 2\left(2\cdot 4 + \frac{1}{2}\cdot 1(4+2)\right) + 3$

$= 25$

Ⓐ 11

Ⓑ 12

Ⓒ 13

Ⓓ 22

Ⓔ 23

25

14. [5 pts] Using the properties of the definite integral find the value of $\int_1^5 f(x)\,dx$ if it is known that $\int_1^6 f(x)\,dx = 4$, $\int_5^6 f(x)\,dx = 1$.

- (A) 1
- (B) 2
- (C) 3
- (D) 4
- (E) 5

$$\int_1^5 f(x)\,dx = \underbrace{\int_1^6 f(x)\,dx}_{=4} + \underbrace{\int_6^5 f(x)\,dx}_{-1 = -\underbrace{\int_5^6 f(x)\,dx}_{=1}}$$

$$= 3$$

15. [5 pts] Compute $\int (x+2)\sqrt{x}\,dx$

(A) $\frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} + C$

(B) $\frac{2}{3}x^{\frac{3}{2}} - \sqrt{x} + C$

(C) $\frac{1}{2}x^2 + 2x + C$

(D) $\frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{3/2} + C$

(E) $\frac{1}{2}x^2 - x^{\frac{3}{2}} + x + C$

$$= \int \left(x\sqrt{x} + 2\sqrt{x}\right) dx$$

$$= \int \left(x^{3/2} + 2x^{1/2}\right) dx$$

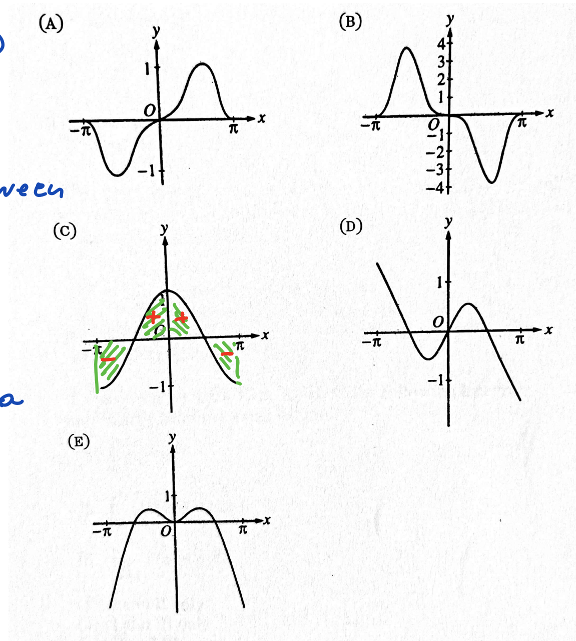
$$= \frac{2}{5}x^{5/2} + 2 \cdot \frac{2}{3}x^{3/2} + C$$

$$= \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + C$$

______ PTS 16. [5 pts] Which function(s), whose graph is below, has a non-zero average value over the closed interval $[-\pi, \pi]$. *[Recall: The average value of a function f on $[a, b]$ is $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$.]*

- $f_{ave} = 0$ in A, B & D as $f(x)$ is an odd function integrated between symmetric bounds.
- $f_{ave} = 0$ in C as there is as much area above the x-axis as there is below

$\therefore$ E is the answer.



Ⓐ A   Ⓒ C   Ⓔ E

Ⓑ B   Ⓓ D

Part II: Free Response

Show all your work neatly and in a structured way.

______PTS 17. [10 pts] (Spring 2022) Use logarithmic differentiation to find the derivative of the function:

$$y = x^{\cos x}. \qquad [*]$$

The final answer must be a function of x. Remember to use correct notation to write your final answer.

Take ln of both sides of [*]: $\ln y = \underbrace{\cos x \ln x}_{= \ln(x^{\cos x})}$ [**]

Differentiate both sides of [**] with respect to x: $\frac{y'}{y} = \cos x \cdot \frac{1}{x} - \sin x \ln x$

In the above equation, isolate y': $\boxed{y' = \left(\cos x \cdot \frac{1}{x} - \sin x \ln x\right) x^{\cos x}}$

______ PTS 18. [10 pts] (Fall 2022) Determine $\frac{dy}{dx}$ given that $y^3 - 4y = x^2e^y$. Express $\frac{dy}{dx}$ as a function of x and y.

Same as above : $3y^2y' - 4y' = x^2e^y y' + 2xe^y$

Isolate y' : $\boxed{y' = \dfrac{2xe^y}{3y^2 - 4 - x^2e^y}}$

_____PTS 19. [10 pts] Compute the following limits:

(a) (Spring 2019) $\lim_{x\to 0} \dfrac{\sin x - x}{1-\cos x}$

$\left[\frac{0}{0}\right]$

L'Hôpital

$$= \lim_{x\to 0} \frac{\cos x - 1}{\sin x}$$

$\left[\frac{0}{0}\right]$

L'Hôpital again

$$= \lim_{x\to 0} \frac{-\sin x}{\cos x}$$

$\left[\frac{0}{1}\right]$

$$= 0 \,//$$

(b) (Spring 2023) $\lim_{x \to 0} \frac{e^{2x} - 2}{e^x + 1}$

$\left[\frac{-1}{2}\right]$

$= -\frac{1}{2}$ //

______PTS 20. [10 pts] Use a linear approximation (or differentials) to estimate $e^{-0.02}$.

$$e^{-0.02} \approx e^{0} + (e^{x})'\Big|_{x=0} (-0.02 - 0)$$

$$\approx 1 + 1(-0.02)$$

$$\approx 0.98 \,//$$

_____ PTS 21. [10 pts] The table below shows the speed $v(t)$ of a runner at a given time t. Time is given in seconds and speed in meters per second.

t [s]	v(t) [m/s]
0	0
0.5	4.58
1	6.81
1.5	7.90
2	8.31
2.5	9.22
3	10.14

(a) Using correct units, interpret the meaning of $\int_{0.5}^{2} v(t)dt$.

(b) Use a left Riemann sum with the three subintervals $[0.5, 1]$, $[1, 1.5]$,and $[1.5, 2]$ to approximate $\int_{0.5}^{2} v(t)dt$.

a) $\int_{0.5}^{2} v(t)\,dt$ represents the distance in meters that the runner covered between 0.5 s and 2 s.

b) $\int_{0.5}^{2} v(t)\,dt \approx 0.5\left[4.58 + 6.81 + 7.90\right] = 9.645$

______PTS 22. [10 pts] Given $f(x) = \dfrac{e^{x^2+1}}{x\arctan(x)}$, what is $f'(x)$?

$$f'(x) = \frac{x\tan^{-1}(x)\cdot 2x\,e^{x^2+1} - e^{x^2+1}\left(x\cdot\frac{1}{1+x^2} + \tan^{-1}(x)\right)}{\left(x\tan^{-1}(x)\right)^2}$$

$$= \frac{e^{x^2+1}\left(2x^2\tan^{-1}x - \frac{x}{1+x^2} - \tan^{-1}x\right)}{\left(x\tan^{-1}(x)\right)^2}$$

______PTS 23. [10 pts] The velocity of a particle traveling upwards is given by $v(t) = e^{2t}$ feet/second. If the height of the particle is 3 feet at $t = 1$ second, at what time is the height 6 feet?

$$h(t) = \int v(t)\, dt$$

$$= \int e^{2t}\, dt$$

$$= \frac{e^{2t}}{2} + C \qquad [*]$$

initial condition: When $t=1$, $h = 3$. Plugging it into [*]:

$$3 = \frac{e^2}{2} + C \Rightarrow C = 3 - \frac{e^2}{2}$$

$$h(t) = \frac{e^{2t}}{2} + 3 - \frac{e^2}{2}$$

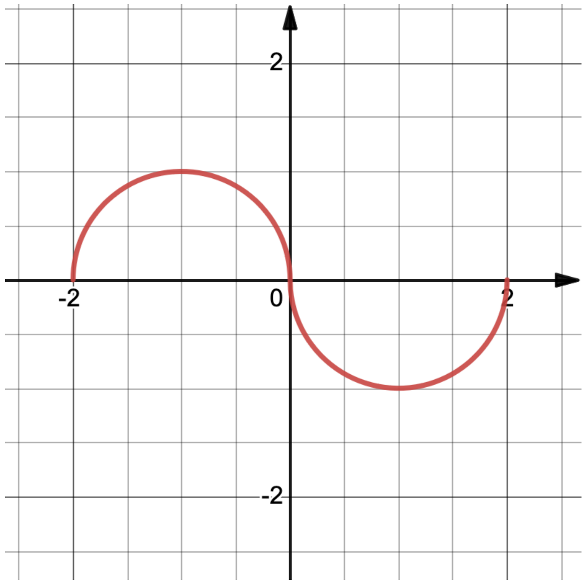
$$6 = \frac{e^{2t}}{2} + 3 - \frac{e^2}{2}$$

$$\Leftrightarrow e^{2t} = 6 + e^2$$

$$\Leftrightarrow 2t = \ln(6+e^2)$$

$$\Rightarrow \boxed{t = \frac{\ln(6+e^2)}{2} \quad (\text{sec})}$$

_____ PTS 24. [10 pts] The graph of $y = f'(x)$, the derivative function of $f(x)$, is shown below. Assume that $f(x)$ is defined and continuous on $[-2, 2]$. Give a complete answer to the following questions.



(a) (2pts) What are the x coordinates of all critical points of $f(x)$?
– If there aren't any, write NONE.

ANSWER: $x = -2, 0, 2.$

(b) (2pts) What are the x coordinates of all local minima of $f(x)$?
– If there aren't any, write NONE.

ANSWER: NONE

(c) (2pts) What are the intervals where $f(x)$ is concave up?
– If there aren't any, write NONE.

ANSWER: $(-2, -1), (1, 2)$

(d) (2pts) What are the x coordinates of all inflection points of $f(x)$?
– If there aren't any, write NONE.

ANSWER: $x = -1, 1$

(e) (2pts) What are the intervals where $f(x)$ is increasing?
– If there aren't any, write NONE.

ANSWER: $(-2, 0)$

______PTS 25. [10 pts] Compute $\int \frac{\pi^2 \sin^2 \pi x}{\sec \pi x}\, dx$.

let $u = \sin(\pi x)$

$du = \pi \cos(\pi x)\, dx$

$$= \int \pi u^2 \, du$$

$$= \pi \frac{u^3}{3} + C$$

$$= \frac{\pi}{3} \sin^3(\pi x) + C \quad //$$

______PTS 26. [10 pts] What is the absolute maximum and the absolute minimum of the function $f(x) = x^4 - 4\pi^3 x + 1$ on the interval $[-1, 4]$?

$f'(x) = 4x^3 - 4\pi^3 = 0$ at $\underbrace{x = \pi}_{\text{critical value}} \in [-1, 4]$

$f(-1) = 2 + 4\pi^3$ ← absolute maximum

$f(\pi) = 1 - 3\pi^4$ ← absolute minimum

$f(4) = 257 - 16\pi^3$

$f(x)$ has an absolute minimum of $1 - 3\pi^4$ at $x = \pi$ and absolute maximum of $2 + 4\pi^3$ at $x = -1$.

______PTS 27. [10 pts] Find the derivative of the function:

$$f(x) = \frac{x(x-5)^{x-2}}{\sin^5(x)}$$

The final answer must be a function of x. Remember to use correct notation to write your final answer.

⚠ Here one can either write $f(x) = e^{\ln(f(x))}$ and differentiate or use log differentiation.
Using the latter:

$$\ln f(x) = \ln x + (x-2)\ln(x-5) - 5\ln(\sin x)$$

$\frac{d}{dx}$

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{x-2}{x-5} + \ln(x-5) - 5\frac{\cos x}{\sin x}$$

$$\boxed{f'(x) = \left(\frac{1}{x} + \frac{x-2}{x-5} + \ln(x-5) - 5\cot(x)\right)\cdot\frac{x(x-5)^{x-2}}{\sin^5(x)}}$$

______PTS 28. [10 pts] What is the function $\phi'(x)$, if $\phi(x) = \int_{x+1}^{x^2} \frac{\sin t}{t^2+1}\, dt$?

$$\phi(x) = \int_{x+1}^{1} \frac{\sin t}{t^2+1}\, dt + \int_{1}^{x^2} \frac{\sin t}{t^2+1}\, dt$$

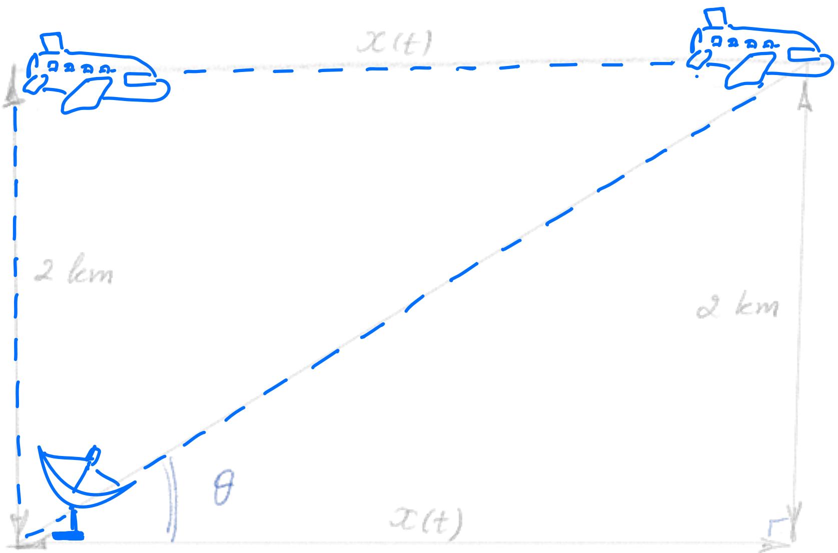
$$= -\int_{1}^{x+1} \frac{\sin t}{t^2+1}\, dt + \int_{1}^{x^2} \frac{\sin t}{t^2+1}\, dt$$

By the Fundamental Theorem of Calculus:

$$\boxed{\phi'(x) = -\frac{\sin(x+1)}{(x+1)^2+1} + \frac{\sin(x^2)}{x^4+1}\cdot 2x}$$

______PTS 29. [10 pts] A plane flies horizontally at an altitude of 2 km and passes directly over a radar located on the ground. When the angle of elevation is 60^o, this angle is decreasing at a rate of 30^o/min. How fast is the plane at that time?

⚠ Everything should be in radians; otherwise the derivative of the trigonometric functions is <u>not</u> correct!



Data:

$$\frac{d\theta}{dt} = -\frac{\pi}{6}\left[\frac{rad}{min}\right] \text{ when } \theta = \frac{\pi}{3}\,[rad]$$

Question:

What is $\frac{dx}{dt}$ when $\theta = \frac{\pi}{3}\,[rad]$?

Let $x(t)$ be the horizontal distance covered by the plane at time t.

$$x(t) = \frac{2}{\tan(\theta(t))} = 2\cot(\theta(t))$$

$\frac{d}{dt}$

$$x'(t) = -2\csc^2(\theta(t))\cdot\theta'(t)$$

When $\theta = \frac{\pi}{3}$ and $\theta' = -\frac{\pi}{6}$

$$x'(t) = -2\cdot\csc^2\left(\frac{\pi}{3}\right)\cdot\left(-\frac{\pi}{6}\right)$$

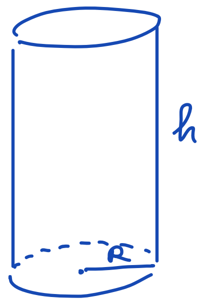
$$= \cancel{-2}\cdot\frac{4}{3}\cdot\left(-\frac{\pi}{\cancel{6}_{3}}\right)$$

$$= \frac{4\pi}{9}\quad[km/min]$$

$$\boxed{x'(t) = \frac{4\pi}{9}\ km/min}$$

_____ PTS 30. [10 pts] Dawg Food Inc. wishes to design a can of a volume of 100 cm^3 using the least amount of metal as possible. The can has to be cylindrical with a lid and a bottom. Find the dimensions it should use.

(a) (2pts) Sketch the can, label the height with h and the radius with R.



$$V = 100\,cm^3 = \pi R^2 h$$

(b) (4pts) Express the exterior area $A(h)$ (in cm^2) of the can in terms of h, **and** give a reasonable domain for A.

$$A = 2A\left(\bigcirc_{R}\right) + A\left(\underset{2\pi R}{\boxed{\qquad}}\ h\right)$$

$$= 2\pi R^2 + 2\pi R h$$

$$= 2\pi\left(\frac{100}{\pi h}\right) + 2\pi \cdot \frac{10}{\sqrt{\pi h}} \cdot h = \frac{200}{h} + 20\sqrt{\pi h}, \quad dom(A) = (0, \infty)$$

(c) (4pts) Find the value of h for which the can requires a minimal amount of metal.

$$A' = -\frac{200}{h^2} + 10\sqrt{\pi} \cdot \frac{1}{\sqrt{h}}$$

$$= \frac{-200 + 10\sqrt{\pi}\, h^{3/2}}{h^2}$$

$= 0 \quad h = \sqrt[3]{\frac{400}{\pi}}$

DNE $\quad h = 0$

Now we need to verify that the critical point that we've found, $h = \sqrt[3]{\frac{400}{\pi}}$, represents a minimum.

We can do that by studying the sign of A'

h	$(0$	$\left(\frac{400}{\pi}\right)^{1/3}$	$\infty)$
sign of A'	$-$	0	$+$

which shows that the minimum of A is reached at $\boxed{h = \sqrt[3]{\frac{400}{\pi}}}$.