UNIVERSITY OF GEORGIA

DEPARTMENT OF MATHEMATICS
MATH 2250 - FINAL EXAM
FALL 2023

PRINTED NAME : ______________________
STUDENT ID : ______________________
DATE : ____/____/____

GRADE
/150

INSTRUCTOR : ______________________
CLASS TIME : ______________________

INSTRUCTIONS

№	SCORE	MAX
1		5
2		5
3		5
4		5
5		5
6		5
7		5
8		5
9		5
10		5
11		5
12		10
13		5
14		10
15		10
16		5
17		5
18		5
19		5
20		5
21		5
22		10
23		10
24		10
TOTAL		150

- The exam lasts 3 hours and it has two parts: the first one consists of Multiple Choice (MC) questions, and the second part of Free Response (FR) ones. You must show work for both parts. An unjustified answer will receive no credit. If you are using a shortcut, explain it.

- Your work must be neat and organized. Circle the answer for MC questions and put a box around the final answer for the FR questions. There is only one correct answer for each MC question.

- Smart devices (including smart watches and cell phones) are not allowed and may not be on your person.

- If you plan to use a calculator, you are only allowed to use a TI-30XS Multiview calculator; the name must match exactly. No other calculators or sharing of calculators is allowed. Include an exact answer for each problem. Answers containing symbolic expressions such as cos(3) and ln(2) are perfectly acceptable.

- If you need extra space, use the last page. Any solution that is without indication on the scrap paper and not in the designated space, will not be graded.

Part I: Multiple Choice

Show your work and circle your answer.

______PTS 1. [5 pts] Let $f(x) = x^2e^x$. What is the value of $f'(1)$?

- (A) e
- (B) $2e$
- (C) $3e$
- (D) $2 + e$
- (E) None of those

$f'(x) = 2x\,e^x + x^2e^x$

$f'(1) = 2 \cdot 1 \cdot e^1 + 1^2 \cdot e^1$

$= 3e$

______PTS 2. [5 pts] Find the limit, if it exists. Otherwise, choose DNE.

$$\lim_{x \to 0} \frac{x^2 - 3}{\cos(5x) + x} \overset{\left[\frac{-3}{1}\right]}{=} -3$$

- (A) DNE
- (B) -3
- (C) 0
- (D) $\frac{1}{5}$
- (E) $\frac{2}{5}$

______PTS 3. [5 pts] Given that $\lim_{x \to 1} \dfrac{f(x) - 4}{2x - 2} = 7$, what is $\lim_{x \to 1} f(x)$?

- (A) 0
- (B) 4
- (C) 7
- (D) 11
- (E) does not exist

_____PTS 4. [5 pts] The graph of $y = \dfrac{x-3}{x^2+4x-21}$ has $= \dfrac{x-3}{(x-3)(x+7)} = \dfrac{1}{x+7}$ for $x \neq 3$

(A) a vertical asymptote at $x = -7$ and a removable discontinuity at $x = 3$.

(B) a vertical asymptote at $x = 3$ and a removable discontinuity at $x = -7$.

(C) removable discontinuities at both $x = -7$ and $x = 3$.

(D) vertical asymptotes at both $x = -7$ and $x = 3$.

(E) neither removable discontinuities nor vertical asymptotes.

_____PTS 5. [5 pts] Let $f(x) = 2x^3 - x^2 + 1$. The tangent line to the graph of $f(x)$ at $x = 1$ is parallel to which of the following lines?

(A) $y = 5x - 1$

(B) $y = 4x + 3$

(C) $y = 2x + 2$

(D) $y = 4$

(E) None of those

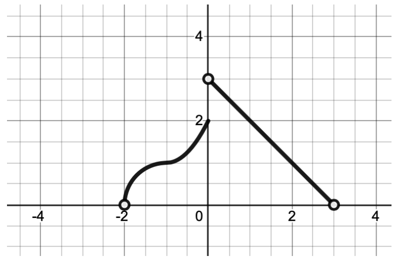
$f'(x) = 6x^2 - 2x$

$f'(1) = 6 \cdot 1^2 - 2 \cdot 1$

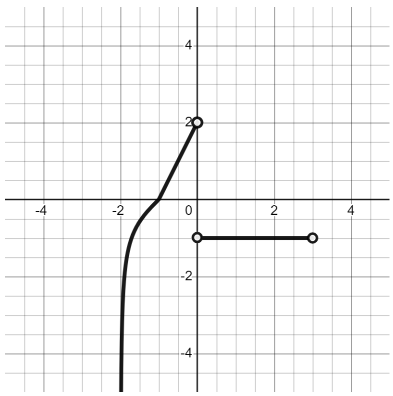
$= 4$

Parallel lines have equal slopes, hence (B) is the answer.

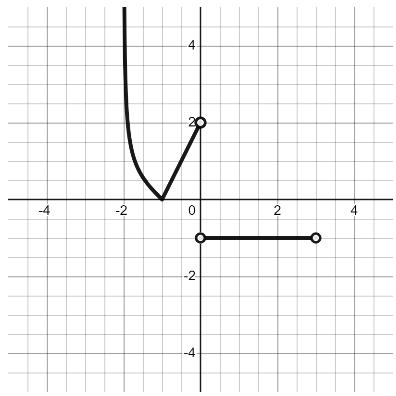
______ PTS 6. [5 pts] The function f is defined on $(-2, 3)$. Its graph is given below:



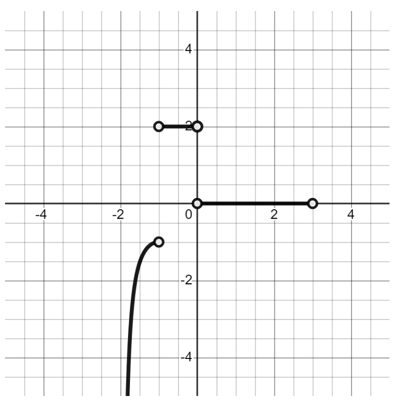
Below are four graphs. One of them is the graph of f', and one of them is the graph of f''.



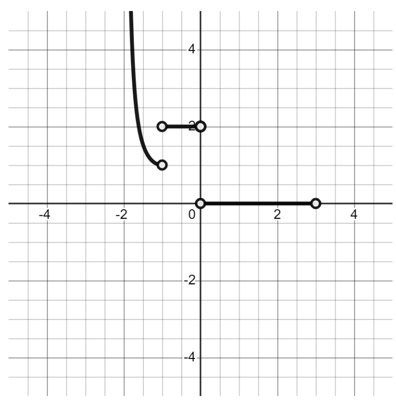
(1)



(2)



(3)



(4)

Which is the graph of f', and which is the graph of f''?
In the choices below, the first number corresponds to the graph of f', the second one to that of f''.

- (A) 1,3
- (B) 3,1
- (C) 1,4
- (D) 2,3
- (E) 3,2

____PTS 7. [5 pts] Given $f(x) = \dfrac{\ln(x^3+x+1)}{x\arcsin(x)}$, what is $f'(x)$?

(A) $\dfrac{\ln(3x^2+1)x\arcsin(x) - \ln(x^3+x+1)x\arccos(x)}{(x\arcsin(x))^2}$

(B) $\dfrac{\ln(x^3+x+1)x\arccos(x) - \ln(3x^2+1)x\arcsin(x)}{(x\arcsin(x))^2}$

(C) $\dfrac{\ln(x^3+x+1)(\arcsin(x)+x\arccos(x)) - \ln(3x^2+1)x\arcsin(x)}{(x\arcsin(x))^2}$

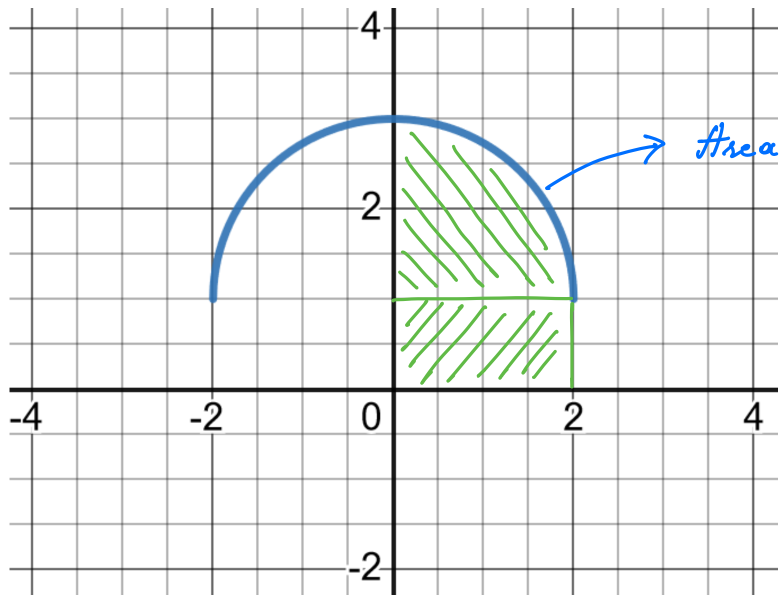
(D) $\dfrac{\left(\frac{3x^2+1}{x^3+x+1}\right)x\arcsin(x) - \ln(x^3+x+1)\left(\arcsin(x)+x\cdot\frac{1}{\sqrt{1-x^2}}\right)}{(x\arcsin(x))^2}$

(E) $\dfrac{\ln(x^3+x+1)\left(\arcsin(x)+x\cdot\frac{1}{\sqrt{1-x^2}}\right) - \left(\frac{3x^2+1}{x^3+x+1}\right)x\arcsin(x)}{(x\arcsin(x))^2}$

$$f'(x) = \frac{x\sin^{-1}(x)\left[\ln(x^3+x+1)\right]' - \ln(x^3+x+1)\cdot\left(x\sin^{-1}(x)\right)'}{(x\sin^{-1}(x))^2}$$

$$= \frac{x\sin^{-1}(x)\cdot\dfrac{3x^2+1}{x^3+x+1} - \ln(x^3+x+1)\cdot\left(x\cdot\dfrac{1}{\sqrt{1-x^2}} + \sin^{-1}(x)\right)}{(x\sin^{-1}(x))^2}$$

________ PTS 8. [5 pts] What is the value of $\int_0^2 \left(1 + \sqrt{4 - x^2}\right) dx$? (Hint: Use the following graph of $y = 1 + \sqrt{4 - x^2}$, which is defined over $[-2, 2]$.)



$$\text{Area} = \frac{1}{4}\pi \cdot 2^2 + 2 \cdot 1$$
$$= \pi + 2$$

(A) π

(B) 2π

(C) 4π

(D) $\pi + 2$

(E) $2\pi + 4$

________ PTS 9. [5 pts] What is $g'(e)$, if $g(x) = \int_1^{x^2} (t \ln t)\, dt$?

(A) 0

(B) 1

(C) e^2

(D) $2e^3$

(E) $4e^3$

$$g'(x) = 2x \cdot x^2 \ln(x^2)$$
$$g'(e) = 2e \cdot e^2 \underbrace{\ln(e^2)}_{=2}$$
$$= 4e^3$$

______PTS 10. [5 pts] Compute $\int \left(x^3 - 8x + x^{2/3}\right) dx.$ $= \frac{x^4}{4} - 4\cancel{8}\frac{x^2}{\cancel{2}} + \frac{3}{5} x^{2/3} + C$

(A) $\frac{1}{4}x^4 - 4x^2 + \frac{3}{5}x^{5/3} + C$

(B) $3x^2 - 8 + \frac{2}{3}x^{-1/3} + C$

(C) $x^3 - 8x + x^{2/3} + C$

(D) $3x^2 - 4x^2 + x^{2/3} + C$

(E) $\frac{1}{4}x^4 - 8x + \frac{3}{5}x^{5/3} + C$

______PTS 11. [5 pts] Using the properties of the definite integral find the value of

$$\int_3^7 (1 - 5f(x))\, dx$$

if it is known that

$$\int_3^8 f(x)\, dx = 10 \quad \text{and} \quad \int_7^8 f(x)\, dx = 8.$$

(A) -46

(B) -10

(C) -9

(D) -6

(E) 2

$$\int_3^7 (1 - 5 f(x))\, dx = \int_3^7 1\, dx - 5 \int_3^7 f(x)\, dx$$

$$= \int_3^7 1\, dx - 5 \left[\int_3^8 f(x)\, dx + \int_8^7 f(x)\, dx \right]$$

$$= \int_3^7 1\, dx - 5 \left[\int_3^8 f(x)\, dx - \int_7^8 f(x)\, dx \right]$$

$$= 4 - 5[10 - 8]$$

$$= -6 \;//$$

Part II: Free Response

Show all your work neatly and in a structured way.

12. [10 pts] Find the following limits. If they do not exist, choose DNE.

______PTS (a) (5pts) $\lim_{x\to 2} \frac{4x^2 - 16}{x - 2}$

$$\left[\tfrac{0}{0}\right] = \lim_{x\to 2} \frac{4\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$

$$= \lim_{x\to 2} 4(x+2)$$

$$= 16 \;//$$

______PTS (b) (5pts) $\lim_{x\to+\infty} x - \sqrt{x^2 + x}$?

$$[\infty - \infty] = \lim_{x\to+\infty} x - \sqrt{x^2+x} \cdot \frac{x + \sqrt{x^2+x}}{x+\sqrt{x^2+x}}$$

$$= \lim_{x\to+\infty} \frac{\cancel{x^2} - (\cancel{x^2}+x)}{x+\sqrt{x^2+x}}$$

$$= \lim_{x\to+\infty} \frac{-\cancel{x}}{\cancel{x}\left(1+\sqrt{1+\frac{1}{x}}\right)} \quad \text{as we consider } x>0.$$

$$= \lim_{x\to+\infty} \frac{-1}{1+\sqrt{1+\frac{1}{x}}}$$

$$= -\frac{1}{2} \;//$$

_______PTS 13. [5 pts] Given the values of $f(x)$ and $f'(x)$ in the table below, and given that

$$h(x) = f(3 + f(x)),$$

find $h'(1)$.

x	$f(x)$	$f'(x)$
1	1	-2
2	4	-1
3	-3	5
4	2	$1/2$

$h'(x) = f'(3 + f(x)) \cdot f'(x)$

$h'(1) = f'(3 + f(1)) \cdot f'(1)$

$= f'(3 + 1) \cdot (-2)$

$= \frac{1}{2} \cdot (-2)$

$= -1$ //

_______PTS 14. [10 pts]

(a) (5 pts) Find the linear approximation of $h(x) = \sqrt{x}$ at $x = 9$.

$L(x) = h(9) + h'(9)(x-9)$ where $h'(x) = \frac{1}{2\sqrt{x}}$

$= 3 + \frac{1}{6}(x-9)$

$= \frac{1}{6}x + \frac{3}{2}$

(b) (5 pts) Use the above to estimate $\sqrt{9.1}$.

$\sqrt{9.1} \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9)$

$= 3 + \frac{0.1}{6}$

$= 3 + \frac{1}{60}$

$= \frac{181}{60}$

____ PTS 15. [10 pts] Find the slope of the tangent to the curve implicitly defined by the equation

$$y^4 - xy^2 + x^4 = 1$$

at the point $(1, 1)$.

$$4y^3y' - 2xyy' - y^2 + 4x^3 = 0$$

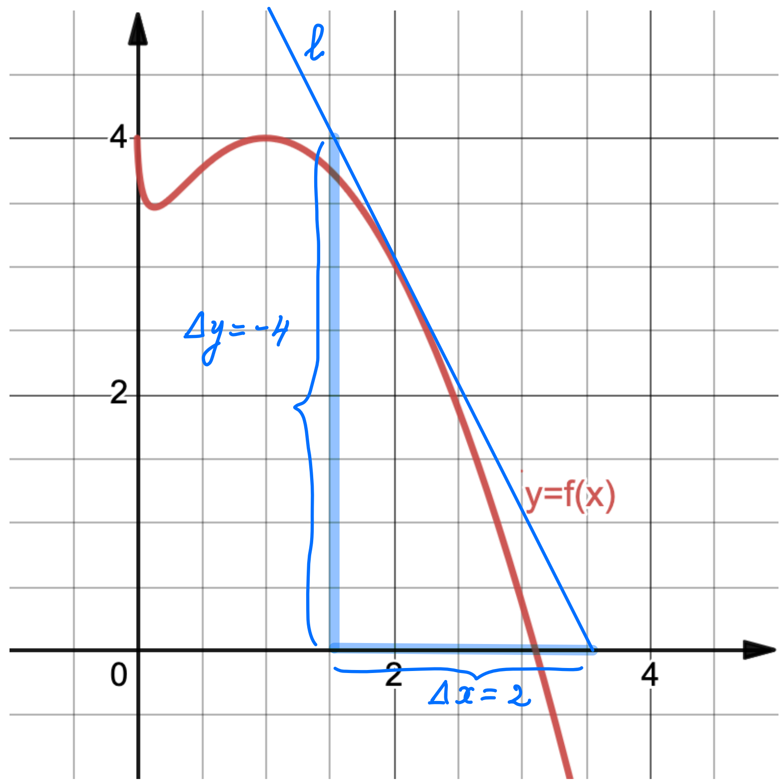
At $(1,1)$:

$$4y' - 2y' - 1 + 4 = 0$$

$$2y' = -3$$

$$y' = -\frac{3}{2} \quad //$$

______PTS 16. [5 pts] Sketched below is the graph of the function $y = f(x)$.



(a) (2pts) On the graph, sketch the tangent line to $f(x)$ at $x = 2$.

(b) (3pts) Use the graph to estimate the value of $f'(2)$. Show the work that leads to your estimate.

$f'(2) =$ slope of tangent

$\approx \frac{\Delta y}{\Delta x}$ of line ℓ

$= -\frac{4}{2}$

$= -2$

______ PTS 17. [5 pts] The derivative of the function $g(x)$ is given below. Both g and g' are defined everywhere except at 0 and 1.

$$g'(x) = \frac{x^2 - 4}{x^2 - x}$$

Answer the following questions.

(a) (3pts) What are the x coordinates of all local minima of $g(x)$?
– If there aren't any, write NONE.

ANSWER: $x = 2$

(b) (2pts) What are the x coordinates of all local maxima of $g(x)$?
– If there aren't any, write NONE.

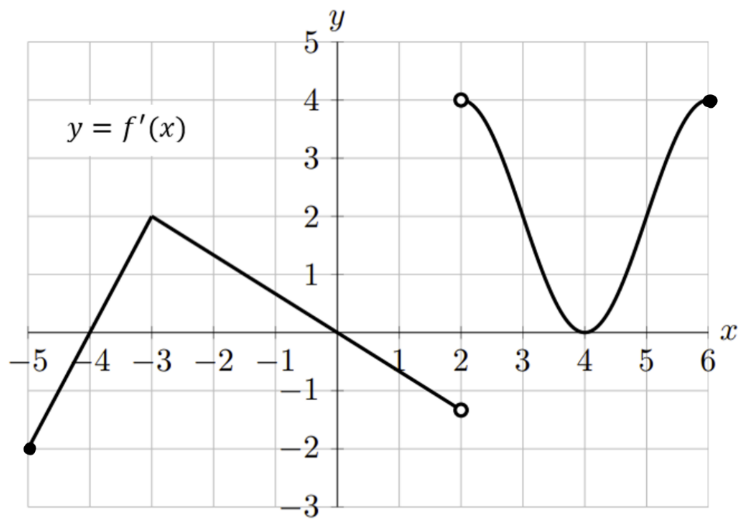
ANSWER: $x = -2$

$$g'(x) = \frac{(x-2)(x+2)}{x(x-1)}$$

Critical values: $x = -2, 2, 0, 1$

x		-2		0		1		2	
sign of $g'(x)$	$+$	0	$-$	//	$+$	//	$-$	0	$+$
		local max						local min	

______ PTS 18. [5 pts] The graph of $y = f'(x)$, the derivative function of $f(x)$, is shown below. Assume that $f(x)$ is defined and continuous on $[-5, 6]$. Give a complete answer to the following questions.



Attention: This is **NOT** the graph of $f(x)$.

(a) (2pts) What are the intervals where $f(x)$ is concave up?
– If there aren't any, write NONE.

ANSWER: $(-5,-3)$, $(4,6)$

(b) (1pt) How many inflection points does the graph of $f(x)$ have?
– If there aren't any, write 0.

ANSWER: 2

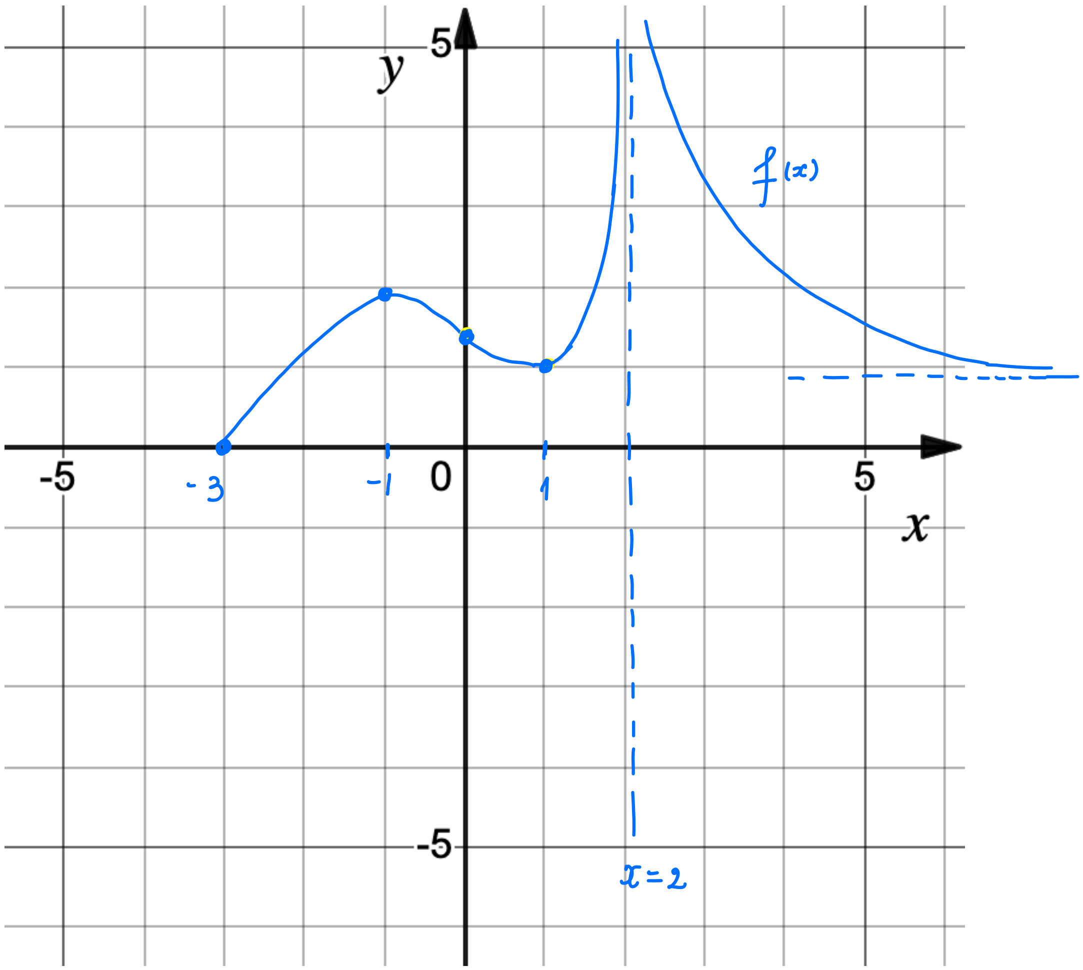
(c) (2pts) What are the intervals where $f(x)$ is increasing?
– If there aren't any, write NONE.

ANSWER: $(-4,0)$, $(2,6)$

_____ PTS 19. [5 pts] Below is the sign chart of a function f whose domain is $[-3,2)\cup(2,\infty)$. Sketch the function as well as possible given the available data.

x	-3	$(-3,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,2)$	2	$(2,+\infty)$	$+\infty$
f	0		loc max				loc. min		$\lim_{x\to 2} f(x) = +\infty$		$\lim_{x\to+\infty} f(x) = 1$
f'	+	+	0	-	-	-	0	+	▩	-	
f''	-	-	-	-	0	+	+	+	▩	+	

inflection pt.



______PTS 20. [5 pts] What are the absolute maxima and the absolute minima of the function

$$f(x) = x^3 - 3x^2 + 1$$

on the interval [0, 5]?

$f'(x) = 3x^2 - 6x$

$= 3x(x-2)$

critical values: $x = 0, 2$

$f(0) = 1$

$f(2) = -3$

$f(5) = 51$

$f(x)$ has an absolute maximum of 51 at $x=5$

an an absolute minimum of -3 at $x=2$.

______PTS 21. [5 pts] Compute $\int x^2 \cos\left(x^3 + 3\right) dx$.

let $u = x^3 + 3$

$du = 3x^2 dx$

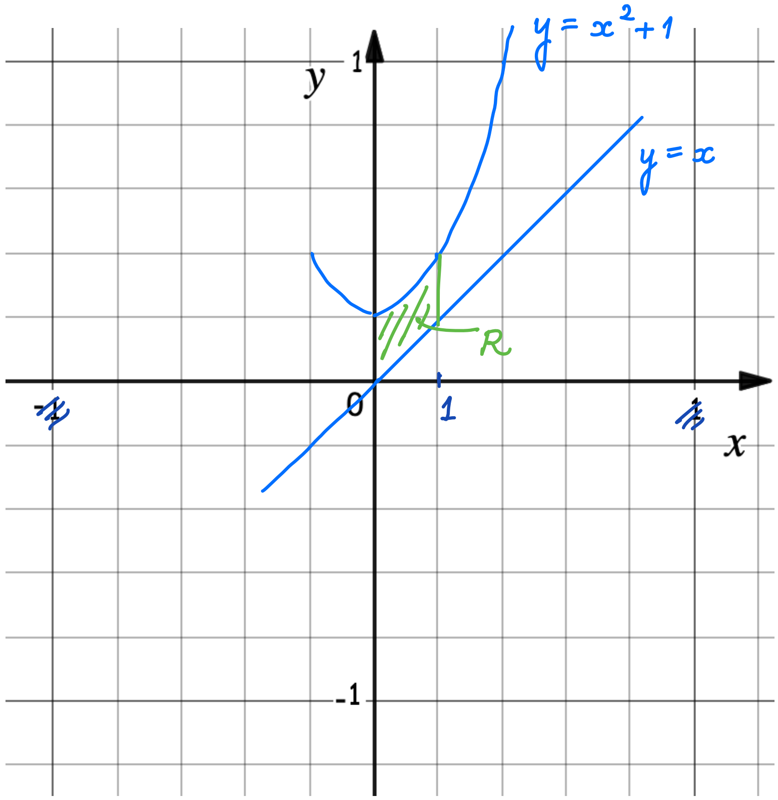
$= \int \cos u \ \frac{1}{3} du$

$= \frac{1}{3} \sin u + C$

$= \frac{1}{3} \sin(x^3+3) + C$ //

______ PTS 22. [10 pts] Consider the region R bounded by $y = x^2 + 1$ and $y = x$ between $x = 0$ and $x = 1$.

(a) (4pts) Sketch R. Make sure to label the graphs.



(b) (6pts) Find its area.

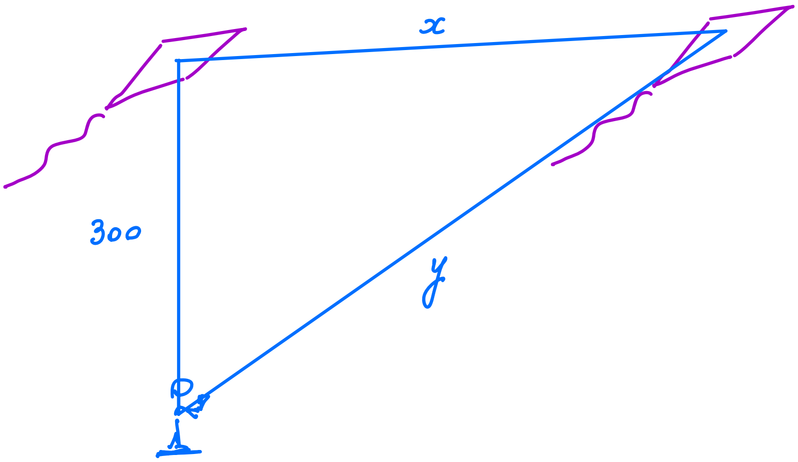
$$A = \int_0^1 (x^2 + 1 - x)\, dx$$

$$= \frac{x^3}{3} + x - \frac{x^2}{2} \Big|_0^1$$

$$= \left[\frac{1}{3} + 1 - \frac{1}{2} \right] - [0]$$

$$= \frac{5}{6}$$

______PTS 23. [10 pts] On a windy day, Camila launches a sunny yellow kite 300 ft into the sky. The wind tugs it horizontally at 25 ft/sec. When the string reaches 500 ft, how fast should she let it out?



$x(t)$ - horizontal distance travelled by the kite

$y(t)$ - distance between the Camila and the tugged kite

Given:

picture & $x'(t) = 25$

Wanted:

$y' = ?$ when $y = 500$

Relation:

$$300^2 + x^2 = y^2$$

$\frac{d}{dt}$

$$\cancel{2}xx' = \cancel{2}yy'$$

$$y' = \frac{xx'}{y}$$

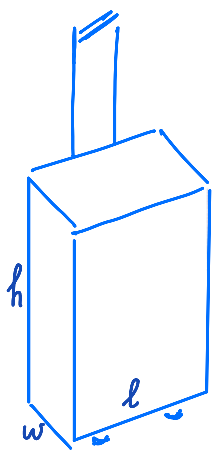
When $y = 500$, $x = 400$ (3, 4, 5) triangle

$$y' = \frac{400 \cdot 25}{500} = 20$$

Answer: When the string reaches 500 ft, Camila lets it out at a speed of 20 ft/sec.

_____ PTS 24. [10 pts] With final exams wrapping up, many people will be traveling over the winter break. In preparation for this, people will need to make sure to check the dimensions of their checked and carry-on luggage to make sure that they are not too big. Passengers of many airlines are only allowed to carry a piece of luggage into an airplane if the total of its length, width, and height does not exceed 45 in.

(a) (4pts) Suppose that you wish to carry on a rectangular suitcase whose length is exactly 1.5 times its width, and whose dimensions add up to 45 in. Let w be the width of the suitcase. Give a formula, $V(w)$, for the volume (in in^3) of such a suitcase in terms of w.



- $l + h + w = 45$ where $l = 1.5w$

$h = 45 - 2.5w$

- $V(w) = lhw$

$= (1.5w)(45 - 2.5w) \cdot w$

$= -3.75w^3 + 67.5w^2$

(b) (2pts) Find a reasonable domain for w.

$w \in (0, \infty)$

(c) (4pts) Find the value of w at which the volume of the suitcase $V(w)$ is maximized.

A is

$V'(w) = -11.25w^2 + 135w$

$= w(-11.25w + 135)$

Critical values: $w = 0, 12$

Now we need to verify that $w = 12$ indeed represents a maximum. We can do this by studying the sign of V'

w	(0	12	∞)
sign of V'	−	0	+

This shows that indeed the maximal volume of such a suitcase is attained when $w = 12$ in.

SCRAP PAPER

Do NOT tear this page off!

SCRAP PAPER

Do NOT tear this page off!